FULLY DISCRETE GALERKIN APPROXIMATIONS OF PARABOLIC BOUNDARY VALUE PROBLEMS WITH NONSMOOTH BOUNDARY DATA

By

GILBERT KAISER CHOUDURY

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1987

ACKNOWLEDGEMENT

I express my sincere gratitude to my advisor Dr. Irena Lasiecka for her guidance and support which have inspired me and without which this work would not have been possible. I would also like to thank the other members of my committee, in particular Dr. Bruce Edwards, who efficiently took care of many details which were nevertheless crucial.

Finally, special thanks are due to Maya Bordyn whose great skill and cooperation have been most appreciated during a trying period.

TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS.. ii

ABSTRACT.. v

CHAPTERS

I INTRODUCTION.. 1

1.1 Notations and Basic Definition........................ 1
1.2 Orientation and Formulation of the Problem............ 9
1.3 Regularity of the Solution............................ 17
1.4 Main Results for the Parabolic Problem (1.6).......... 19
1.5 Survey of the Literature.............................. 25
1.6 Outline of the Work................................... 31

II SEMIGROUP REPRESENTATION FOR PARABOLIC BOUNDARY-VALUE PROBLEMS 33

2.1 Opening Remarks....................................... 33
2.2 Semigroup Formula for the Solution of an Abstract Boundary-Value Problem................................ 35
2.3 Application to a Canonical Example of a Parabolic Boundary-Value Problem................................ 40
2.4 Regularity of the Solution to (2.1)................... 44

III SPACE DISCRETIZATION... 57

3.1 Opening Remarks....................................... 57
3.2 Galerkin Approximation of an Abstract Boundary-Value Problem.. 58
3.3 Estimation of the Error............................... 61
3.4 Application to a Canonical Parabolic Boundary-Value Problem.. 63

IV TIME DISCRETIZATION... 72

4.1 Opening Remarks....................................... 72
4.2 A One-Dimensional Euler Scheme........................ 73
4.3 Extension of the Scheme to an Abstract Parabolic Boundary-Value Problem................................ 75
4.4 Preliminary Results................................... 77
4.5 Application to a Canonical Example.................... 84

V FULL TIME AND SPACE DISCRETIZATION.......................... 92

5.1 Opening Remarks.. 92
5.2 The Abstract Discretization Scheme..................... 92
5.3 Estimation of the Error................................ 96
5.4 Application to a Canonical Example..................... 101

VI PROOFS OF THEOREMS 3.1, 4.2, 5.1, 5.2, and LEMMA 4.1......... 105

6.1 Opening Remarks.. 105
6.2 Proof of Theorem 3.1................................... 105
6.3 Proof of Lemma 4.1..................................... 113
6.4 Proof of Theorem 4.2................................... 124
6.5 Proof of Theorem 5.1................................... 139
6.6 Proof of Theorem 5.2................................... 150

VII NUMERICAL EXAMPLES... 158

7.1 Opening Remarks.. 158
7.2 Implementation of the Fully Discrete Scheme for the Heat Equation... 158
7.3 Numerical Computations and Results..................... 161

REFERENCES 166

BIOGRAPHICAL SKETCH... 168

Abstract of Dissertation Presented to the Graduate School of
the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

FULLY DISCRETE GALERKIN APPROXIMATIONS OF PARABOLIC BOUNDARY VALUE PROBLEMS WITH NONSMOOTH BOUNDARY DATA

By

Gilbert Kaiser Choudury

December 1987

Chairperson: Dr. Irena Lasiecka
Major Department: Mathematics

Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Consider the following boundary-value problem:

$$(1)\quad \begin{cases} y_t(t,x) = -A(x,\partial)y(t,x) & \text{on } Q = (0,T)x\Omega \\ y(0,x) = y_o(x) & \text{on } \Omega \\ y(t,x) = g(t,x) & \text{on } \Sigma = (0,T)x\Gamma \end{cases}$$

where g is taken in $L_p[0,T;L_2(\Gamma)]$ with $1 \leq p \leq \infty$. The objectives of this study are threefold:

(i) to construct a fully discrete Galerkin approximation of (1);

(ii) for an abstract model containing problem (1) as a special case, to derive error estimates in the norm of $L_p[0,T;L_2(\Omega)]$ for boundary input in $L_p[0,T;L_2(\Gamma)]$, where $1 \leq p \leq \infty$ and

(iii) to present numerical results corroborating the theoretical rates of convergence.

The distinctive feature of this work resides in the nature of the boundary conditions, which are assumed to be in $L_p[0,T;L_2(\Gamma)]$ for $1 \leq p \leq \infty$. It is well known that standard methods based on variational inequalities require a certain smoothness of the boundary input which in the case of Dirichlet boundary conditions is $H^{1/2}(\Gamma)$. It is therefore apparent that the treatment of "rough" Dirichlet boundary data requires the development of a new approach.

Such a new approach is based on semigroup representation of the solution to problem (1), as well as on that of its semidiscrete and discrete approximations, and on the approximation of singular integrals. We make use of this approach to derive theoretical convergence estimates for the fully discrete approximation y_{hk} to the solution y of problem (1). These estimates are shown to be

$$(2) \qquad |y-y_{hk}|_{L_p[0,T;L_2(\Omega)]} \leq c(h^{1/2}\ln\frac{1}{h} + k^{1/4}\ln\frac{1}{k})|g|_{L_p[0,T;L_2(\Gamma)]}$$

for $1 \leq p \leq \infty$, and with the restriction $k \leq ch^2$ for some $c>0$. These estimates are also shown to be optimal in the case $p = 1, \infty$, and almost optimal (modulo $\ln\frac{1}{h}$ and $\ln\frac{1}{k}$) in the case $1 < p < \infty$.

The construction of the fully discrete approximation scheme makes use of a finite element technique for space discretization, combined with a basic finite-difference scheme for the time variable.

The resulting algorithm is then applied to the canonical case of the one-dimensional heat equation and the results of a computer program implementing this algorithm confirm the theoretical estimates stated in inequality (2).

CHAPTER I
INTRODUCTION

1.1 Notations and Basic Definitions

The main goal of this work is the derivation of theoretical estimates for the error associated with the full discretization of a parabolic equation with nonsmooth Dirichlet boundary conditions. In the canonical case of the heat equation, for example, one would consider the discretization of the following boundary-value problem:

$$\begin{aligned} &\frac{\partial y}{\partial t}(t,x) = \Delta y(t,x) && \text{on } Q = (0,T)\times\Omega \\ &y(0,x) = 0 && \text{on } \Omega \\ &y(t,x) = g(t,x) && \text{on } \Sigma = (0,T)\times\Gamma, \end{aligned}$$

where Ω is an open domain in $\mathbb{R}^n$ with smooth boundary Γ, and where g is arbitrarily chosen in $L_2(\Sigma)$.

Before exposing the problem however, we need to introduce the basic spaces and operators in terms of which our problem can be precisely stated and solved, along with some of their properties.

Let $x = (x_1,\ldots,x_n)$ and $dx = dx_1 \ldots dx_n$. We denote by $L_2(\Omega)$ the space of equivalence classes of measurable functions u such that

(1.1) $$\|u\|_{L_2(\Omega)} = (\int_\Omega |u(x)|^2 dx)^{1/2} < \infty$$

With the inner product associated to the norm (1.1), i.e.

$$(u,v)_{L_2(\Omega)} = \int_\Omega u(x)\bar{v}(x)dx,$$

$L_2(\Omega)$ is a Hilbert space.

Before introducing certain Sobolev spaces, we briefly recall the definition of distributions on Ω. We define

$\mathcal{D}(\Omega) = \{\phi$: ϕ is infinitely differentiable on Ω with compact support in $\Omega\}$

Then

$\mathcal{D}'(\Omega)$ = dual of $\mathcal{D}(\Omega)$
= space of distributions on Ω.

$\mathcal{D}'(\Omega)$ is provided with the weak topology, i.e.,

$$f_n \to f \text{ in } \mathcal{D}'(\Omega) \text{ if and only if } \langle f_n,\phi\rangle \to \langle f,\phi\rangle \quad \forall\phi\in\mathcal{D}(\Omega).$$

Remark. Any function in $L_2(\Omega)$ (or in $L_1(\Omega)$ for that matter) defines a distribution on Ω, also denoted by f.

The derivative $\frac{\partial f}{\partial x_i}$ of a distribution is defined by

$$\langle\frac{\partial f}{\partial x_i},\phi\rangle = - \langle f, \frac{\partial \phi}{\partial x_i}\rangle \qquad \forall\phi\,\mathcal{D}\in(\Omega)$$

and is also a distribution. Then $D^\alpha f$ is defined by iteration, where

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \text{ a multi-index and}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

We are now in a position to define some Sobolev spaces. First, let $m \geq 1$, m an integer. Then $H^m(\Omega)$, the Sobolev space on Ω of order m is defined by

$$H^m(\Omega) = \{u: \ D^\alpha u \in L_2(\Omega), \ \forall \alpha, |\alpha| \leq m\}$$

with norm

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L_2(\Omega)} \Big)^{1/2}.$$

$H^m(\Omega)$ is a Hilbert space with the inner product associated with the norm above, i.e.,

$$(u,v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L_2(\Omega)}.$$

The spaces $H^m(\Gamma)$ are defined in a similar way.

The space $H^m_o(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ in the norm of $H^m(\Omega)$. In general $H^m_o(\Omega) \neq H^m(\Omega)$. We will clarify this point later on in this work.

We also define the spaces $L_p[0,T;H]$ for $1 \leq p \leq \infty$, where H is a Hilbert space, as the space of (equivalence classes of) functions f, strongly measurable on [0,T], with range in X and such that

$$\left(\int_0^T \|f(t)\|_H^p \, dt\right)^{1/p} < \infty \quad \text{for } 1 \le p < \infty$$

and

$$\operatorname*{ess\,sup}_{0 \le t \le T} \|f(t)\|_H < \infty \quad \text{for } p = \infty.$$

We note that $L_2[0,T;H]$ is a Hilbert space with the associated inner product

$$(f,g)_{L_2[0,T;H]} = \int_0^T (f(t),g(t))_H dt$$

If $f, \frac{df}{dt}, \ldots, \frac{d^m f}{dt^m}$ all belong to $L_p[0,T;H]$, then f belongs by definition to $H_p^m[0,T;H]$, with norm

$$\|f\|_{H_p^m[0,T;H]} = \sum_{j=o}^{m} \left\|\frac{d^j f}{dt^j}\right\|_{L_p[0,T;H]}.$$

Finally, we define the spaces $H_p^{r,s}(Q)$, where $Q = (0,T)x\Omega$, r and s are nonnegative integers, as

$$H_p^{r,s}(Q) = L_p[0,T;H^r(\Omega)] \cap H_p^s[0,T;L_2(\Omega)],$$

with norm

$$\|u\|^2_{H_p^{r,s}(Q)} = \|u\|^2_{L_p[0,T;H^r(\Omega)]} + \|u\|^2_{H_p^s[0,T;L_2(\Omega)]}.$$

The spaces $H_p^{r,s}(\Sigma)$, where $\Sigma = (0,T)x\Gamma$ are defined in the same way.

Essential to the forthcoming study is the notion of intermediate or interpolation space, of which we recall the definition [13, Vol. I].

Let X,Y be two Hilbert spaces with

(1.2) $X \subseteq Y$, X dense in Y,

with continuous injection, i.e.,

(1.3) $|x|_Y \leqq c\ |x|_X \quad \forall x \in X.$

Let $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ be the scalar products in X and Y, respectively. Let D(S) be the set of u's such that the antilinear form

$$v \to (u,v)_X,\ v \in X$$

is continuous in the topology induced by Y, i.e.,

$$|v_n|_Y \to 0 \implies (u,v_n)_X \to 0 \quad \forall u \in D(S).$$

Then

(1.4) $(u,v)_X = (Su,v)_Y$

defines S as an unbounded operator in Y with domain D(S).

We note that

(i) D(S) is dense in Y.

In fact, since X is dense in Y, then so is Y' in X' by duality, where the duality is considered with respect to the inner product in X. Therefore, identifying X and its own dual X', we have

$$Y' \subseteq X \subseteq Y,$$

each space being dense in the next larger space. Furthermore, the space D(S) can be identified by its definition with Y', and thus our claim is established.

(ii) S is selfadjoint.

(iii) S is strictly positive, and in fact,

$$(Sv,v)_Y = |v|_X^2 \geq c|v|_Y^2.$$

//

The powers S^θ of S, $\theta \in \mathbb{R}$ can then be defined by using the spectral decomposition of selfadjoint operators. In particular, we define the operator $T: Y \to Y$ by

$$(1.5) \qquad T = S^{1/2}.$$

T is positive and selfadjoint in Y, with domain X. This can be readily seen, as we deduce from (1.4) and (1.5), that

$$(u,v)_X = (Tu,Tv)_Y \qquad \forall u,v \in X.$$

We remark that since S depends on the choice of the scalar products on X and Y, T is dependent on those too, and therefore is not intrinsically linked to X and Y. We are now prepared to define the intermediate spaces $[X,Y]_\theta$.

Definition. Given two Hilbert spaces X and Y satisfying assumptions (1.2) and (1.3), and T defined by (1.5), we set

$$[X,Y]_\theta = D(T^{1-\theta}) \qquad \text{for } 0 \leq \theta \leq 1$$

with the graph norm, i.e.,

$$(|u|_Y^2 + |T^{1-\theta}u|_Y^2)^{1/2}.$$

//

It can be shown that $[X,Y]_\theta$ is intrinsically linked to X and Y, i.e., that it is not dependent on the particular scalar products chosen on X and Y.

Remarks.

(i) from the properties of the spectral decomposition we have that X is dense in $[X,Y]_\theta$.

(ii) $[X,Y]_0 = X$ and $[X,Y]_1 = Y$.

//

Using the method just described, we can define the intermediate spaces $H^s(\Omega)$, $H_p^{r,s}(Q)$ for $r, s \geq 0$.

For example,

$$H^s(\Omega) \overset{\text{def}}{\equiv} [L_2(\Omega), H^1(\Omega)]_s \qquad \text{for } 0 \leq s \leq 1,$$

and similarly for other nonnegative values of s.

Then

$$H_p^{r,s}(Q) \overset{\text{def}}{\equiv} L_p[0,T;H^r(\Omega)] \;\; H_p^s[0,T;L_2(\Omega)] \quad \text{for } r,s \geq 0.$$

Sobolev spaces with negative indices are defined in the following way:

$$H^{-s}(Q) = (H_o^s(\Omega))' = \text{dual of } H_o^s(\Omega), \qquad s \geq 0.$$

If we set

$$H_{p;o,o}^{r,s}(Q) = \overline{\mathcal{D}(Q)} = \text{closure of } \mathcal{D}(Q) \text{ in } H_p^{r,s}(Q),$$

then we can define

$$H_p^{-r,-s}(Q) = (H_{p;o,o}^{r,s}(Q))', \qquad r,s \geq 0.$$

For a bounded operator $A: X \to Y$, the operator norm will be denoted

$$|A|_{X\to Y} = \sup_{|x|_X \leq 1} |Ax|_Y$$

or if there is no ambiguity, simply $|A|$.

The notation $(\cdot,\cdot)$ will be used for the inner product in $L_2(\Omega)$ or $L_2[0,T;L_2(\Omega)]$, the context resolving any ambiguity, and similarly $\langle\cdot,\cdot\rangle$ will be used for the inner product in $L_2(\Gamma)$ or $L_2[0,T;L_2(\Gamma)]$.

We shall sometimes use the notation D_{x_j} in place of $\frac{\partial}{\partial x_j}$, and also $|\cdot|_s$ in place of $|\cdot|_{H^s(\Omega)}$ or $|\cdot|_{H^s(\Gamma)}$, whenever the context resolves any ambiguity.

1.2 Orientation and Formulation of the Problem

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Let $T > 0$ be fixed, and consider the following parabolic problem with nonhomogeneous Dirichlet boundary conditions:

$$(1.6)\quad \begin{cases} \frac{\partial y}{\partial t}(t,x) = A(x,\partial)y(t,x) & \text{on } Q = (0,T)\times\Omega \\ y(0,x) = 0 & \text{on } \Omega \\ y(t,x) = g(t,x) & \text{on } \Sigma = (0,T)\times\Gamma \end{cases}$$

where $g \in L_2(\Sigma)$ is given and

$$(1.7)\quad A(x,\partial)y = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}y) + \sum_{j=1}^{n} b_i(x)D_{x_i}y + c(x)y$$

is a uniformly strongly elliptic, second order differential operator, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} \xi_i^2 \quad \text{for some } \gamma > 0$$

with a_{ij}, b_i, $c \in C^{\infty}(\Omega)$.

We point out that the matrix of coefficients (a_{ij}) is not assumed to be symmetric, which is significant in regard to some of the literature. (See section 1.5 on the literature).

This problem is a special case of a general abstract model which is the framework of our study. In this general context, our aim is to

(i) construct a fully discrete Euler-Galerkin approximation y_{hk} of the solution y to the problem (1.6) (h and k are the parameters of discretization in space and time respectively).

(ii) derive optimal error estimates (with respect to the regularity of the solution y) for $y - y_{hk}$ in $L_p[0,T;L_2(\Omega)]$ norms for problems with "rough" $L_p[0,T;L_2(\Gamma)]$ boundary data.

(iii) present numerical results from a computer program which implements the aforementioned approximation scheme in the canonical case $A(x,\partial) = \Delta$ (Laplacian). These results will be seen to confirm our theoretical estimates.

The first step in the construction of this scheme is to select the basis functions of a finite-dimensional space V_h, which is a subspace of the space of solutions to (1.6). One can then formulate a continuous-time Galerkin approximation of problem (1.6), the solution of which lies in V_h for each t.

The next step is time discretization for which we will use a basic finite difference scheme. This scheme produces a sequence of approximating functions in V_h which approximates the solution at discrete time values. More precisely, one obtains a sequence of linear systems of algebraic equations, the solutions of which are the desired approximating functions. The task of solving these linear systems is conveniently handled by computers, which can produce quick and accurate results.

At this point we emphasize that the low regularity of the boundary data, $L_2(\Sigma)$ or more generally $L_p[0,T;L_2(\Gamma)]$ as opposed to $H^{1/2}(\Sigma)$ boundary data as in certain known treatments in the literature [4,5] (or spaces with higher regularity), is the main source of difficulty. In fact, standard approximation methods [2, 4, 5, 8, 21] are based on the projection of the variational formulation of the solution onto finite-dimensional subspaces. As we shall see later, the variational formulation of the problem (1.6) requires a higher degree of smoothness of the boundary data than is available. Thus an approach different from the variational one has to be developed.

The techniques that we propose to deal with $L_p[0,T;L_2(\Gamma)]$ boundary data are based on semigroup representation of the solution and its approximations. Their presentation is relegated to the next chapter. We note here that this approach is applicable to a variety of situations, for example Neumann boundary conditions with boundary data in the space of distributions $(H^s(\Gamma))'$, $s \geq 0$, or also the case of a differential operator of order $2m$, $m \geq 1$, m an integer. Also of significance is the fact that no restrictions are placed on the nonselfadjointness of the operator $A(x,\partial)$. The only assumption made is that $A(x,\partial)$ with the proper domain be the generator of an analytic semigroup.

The motivation for using the semigroup approach as opposed to standard variational arguments comes precisely, as mentioned before, from the low regularity of the boundary input g. In fact, we shall explain below why the standard variational approach is not adequate to treat problems with "nonsmooth" boundary data.

A standard way of treating the nonhomogeneous Dirichlet boundary conditions is to subtract the effect of the boundary data from the variable and to consider the corresponding nonhomogeneous equation with homogeneous boundary conditions. To this end, we let $\tilde{g}$ be an extension of g into the interior of Ω and we define the new variable $\tilde{y} = y-\tilde{g}$. Then

$$(1.6') \quad \begin{cases} \tilde{y}_t = A(x,\partial)\tilde{y} + A(x,\partial)\tilde{g} - \tilde{g}_t & \text{on } Q \\ \tilde{y}(0,x) = -\tilde{g}(0,x) & \text{on } \Omega \\ \tilde{y}|_\Gamma \equiv 0 & \text{on } \Sigma \end{cases}$$

is the new version of problem (1.6) with homogeneous Dirichlet boundary conditions.

Let $f = A(x,\partial)\tilde{g} - \tilde{g}_t$. In weak form, problem (1.6') becomes

$$(1.8)\quad \begin{cases} (\tilde{y}_t,v)_{L_2(\Omega)} = -\,a(\tilde{y},v)_{L_2(\Omega)} + (f,v)_{L_2(\Omega)} & \text{for } \tilde{y},v \in H_0^1(\Omega) \\ \tilde{y}(0,x) = -\,\tilde{g}(0,x) & \text{on } \Omega \end{cases}$$

where $a(u,v) = -\,(A(x,\partial)u,v)_{L_2(\Omega)}$ is the bilinear form associated to $A(x,\partial)$.

If $\tilde{y}(t)$, $v(t)$ in (1.8) are restricted to lie in some finite-dimensional subspace of $H_0^1(\Omega)$, then (1.8) defines a Galerkin approximation of the solution $\tilde{y}$. The stability of the resulting approximation is proved by using the following standard variational argument based on the coercivity of the bilinear form $a(u,v)$. Let $v = \tilde{y}$. Then

$$(\tilde{y}_t,\tilde{y})_{L_2(\Omega)} = -\,a(\tilde{y},\tilde{y}) + (f,\tilde{y})_{L_2(\Omega)}\,.$$

Since $(\tilde{y}_t,\tilde{y}) = \frac{1}{2}\frac{d}{dt}(\|\tilde{y}\|^2)$, by integrating in t we have

$$\tfrac{1}{2}\int_0^T \frac{d}{dt}(\|\tilde{y}\|^2)dt = \int_0^T -\,a(\tilde{y}(t),\tilde{y}(t))dt + \int_0^T (f(t),\tilde{y}(t))dt$$

and thus

$$\tfrac{1}{2}\left(|\tilde{y}(T)|^2_{L_2(\Omega)} - |\tilde{y}(0)|^2_{L_2(\Omega)}\right) + \int_0^T a(\tilde{y}(t),\tilde{y}(t))dt = \int_0^T (f(t),\tilde{y}(t))dt.$$

By H^1-coercivity of the bilinear form a(u,v), i.e.:

$$a(u,u) \geq c|u|^2_{H^1(\Omega)} - \alpha|u|^2_{L_2(\Omega)} \qquad \forall u \in H^1(\Omega) \text{ for some } c,\alpha>0$$

and by duality (generalized Schwartz inequality), we have

$$\tfrac{1}{2}|\tilde{y}(T)|^2_{L_2(\Omega)} + c\int_0^T |\tilde{y}(t)|^2_{H^1(\Omega)} dt - \alpha\int_0^T |\tilde{y}(t)|^2_{L_2(\Omega)} dt \tag{1.9}$$

$$\leq \int_0^T |f(t)|^2_{H^{-1}(\Omega)} |\tilde{y}(t)|^2_{H^1(\Omega)} dt + \tfrac{1}{2}|\tilde{g}(0)|^2_{L_2(\Omega)}.$$

From the identity

$$2\,ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2, \qquad \varepsilon > 0$$

we obtain

$$\tfrac{1}{2}|\tilde{y}(T)|^2_{L_2(\Omega)} + c\int_0^T |\tilde{y}(t)|^2_{H^1(\Omega)} dt \leq c\int_0^T |\tilde{y}(t)|^2_{L_2(\Omega)} dt + c\cdot\frac{1}{\varepsilon}\int_0^T |f(t)|^2_{H^{-1}(\Omega)} dt$$

$$+ c\cdot\varepsilon\int_0^T |\tilde{y}(t)|^2_{H^1(\Omega)} dt + \tfrac{1}{2}|\tilde{g}(0)|^2_{L_2(\Omega)}.$$

Let $0 < \varepsilon < 1$. Then

$$\frac{1}{2}\|\tilde{y}(T)\|^2_{L_2(\Omega)} + c(1-\varepsilon)\int_0^T \|\tilde{y}(t)\|^2_{H^1(\Omega)}\,dt \tag{1.10}$$

$$\leq \frac{c}{\varepsilon}\int_0^T \|f(t)\|^2_{H^{-1}(\Omega)}\,dt + \frac{1}{2}\|\tilde{g}(0)\|^2_{L_2(\Omega)}\;.$$

In order to conclude that the left-hand side of (1.10) is well-defined, we need to assume that $f \in L_2[0,T;H^{-1}(\Omega)]$. As $f = A(x,\partial)\tilde{g} - \tilde{g}_t$, it follows that we need to assume that $\tilde{g} \in L_2[0,T;H^1(\Omega)]$. A basic trace theorem [13, Vol. I, p. 39] enables us to conclude that $g(t)$ must be taken at least in $H^{1/2}(\Gamma)$.

It is now clear that this standard approach based on H^1-coercivity of the bilinear form $a(u,v)$ is inadequate for treating boundary input in $L_p[0,T;L_2(\Gamma)]$, and therefore a new approach is necessary. As we have indicated before, we will successfully use the semigroup approach combined with duality theory to treat this type of nonsmooth boundary conditions.

Our techniques are based on the approximation of a semigroup formula and on results from the theory of singular integrals [20] applied to the following abstract initial-value problem on a Hilbert space H with input from a Hilbert space U:

$$(1.11)\quad \begin{cases} u_t = -Au + Bf & \text{on } D(A^*)' \;{}^{(\dagger)} \\ u(0) = u_o \end{cases}$$

where $f \in L_p[0,T;U]$, $B: U \to D(A^*)'$ is an operator such that $D(B)$ is possibly empty but $A^{-1}B \in \mathcal{L}(U \to H)$, and $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup $S(t)$. The solution u --understood as a weak solution--then satisfies the formal semigroup formula

$$(1.12)\quad u(t) = S(t)u_o + \int_0^t S(t-\tau)Bf(\tau)d\tau.$$

We point out that the kernel $S(t-\tau)B$ inside the integral in (1.12) is singular and behaves "comparably" to the kernel of the Hilbert transform. In fact,

$$|S(t-\tau)B|_{U\to H} = |AS(t-\tau)A^{-1}B|_{U\to H} \le c|AS(t-\tau)|_{H\to H}$$

since $A^{-1}B \in \mathcal{L}(U \to H)$. From the analyticity of $S(t)$, it follows that

$$|AS(t-\tau)|_{H\to H} \le \frac{c}{t-\tau}$$

Therefore the integral in (1.12) is to be considered as a singular integral defined from U into H. In Section 2.2 it will be shown that

---

(†)The solution of (1.11) should be understood as a weak solution, i.e., in the sense of distributions:

$$(u_t,\phi)_H = -(Au,\phi)_H + (Bf,\phi)_H \quad \forall\phi\in D(A^*)$$

(1.12) defines an element $u(t) \in H$ for almost every $t \in [0,T]$, i.e., a strong solution.

In Lasiecka [11], the continuous-time approximation u_h is achieved by using Galerkin approximations of the above singular integral. Our goal in this work is to extend the techniques of Lasiecka [11, 12] to the discrete-time approximation u_k and to the fully discrete approximation u_{hk} (with discrete semigroups). By doing so we will be able to prove stability and the optimal rates of convergence of the resulting fully discrete scheme (see Theorems 5.1, 5.2).

This general abstract approach will then be applied to the following canonical parabolic boundary-value problem

$$(1.13)\quad \begin{cases} y_t = A(x,\partial)y + f & \text{on } Q \\ y(0,x) = y_o(x) & \text{on } \Omega \\ y|_\Gamma \equiv g & \text{on } \Sigma \end{cases}$$

where $g \in L_p[0,T;L_2(\Gamma)]$, $y_o \in H^{-1/2}(\Omega)$, $f \in H_p^{-3/2,-3/4}(Q)$, and $A(x,\partial)$ is a uniformly strongly elliptic, second order differential operator given by (1.7). The application of abstract convergence results will then produce corresponding convergence results (with the rates of convergence) for the parabolic problem (1.13) (see Theorems 3.3, 4.4, 5.3, 5.4).

1.3 Regularity of the Solution

In order to appreciate results on the rates of convergence of numerical approximations, it is necessary to understand well the behavior of the original continuous solution and in particular the optimal regularity of the solution. It is the purpose of this section to present some of the regularity results which are pertinent to our further analysis.

We shall start by recalling the definition of a strong solution to (1.13).

Definition. A function $y \in L_2(Q)$ is said to be a strong solution of (1.13) if there is a sequence $\{y_n\}$ in $\mathcal{D}(\overline{Q})$ such that

$$|y_n - y|_{L_2(Q)} \to 0$$

$$|y_n - g|_{L_2(\Sigma)} \to 0$$

$$|D_t y_n - A(x,\partial)y_n - f|_{L_2(Q)} \to 0$$

$$|y_n(0) - y_o|_{L_2(\Omega)} \to 0 \qquad \text{as } n \to \infty.$$

//

The following basic result on well-posedness and regularity for (1.13) is recalled from Lions-Magenes [13, Vol. II, p. 78].

Theorem 1.1 [13]. For any given $g \in L_2(\Sigma)$, $y_o \in H^{-1/2}(\Omega)$, $f \in H^{-3/2,-3/4}(Q)$, there is a unique strong solution y to problem (1.13). Moreover,

$$|y|_{H^{1/2,1/4}(Q)} \leq c|g|_{L_2(\Sigma)} + c|y_o|_{H^{-1/2}(\Omega)} + c|f|_{H^{-3/2,-3/4}(Q)}.$$

//

In the sequel we shall need a regularity result of a similar nature to Theorem 1.1, but with the more general data $g \in L_p[0,T;L_2(\Gamma)]$ for $1 \leq p \leq \infty$.

Below we provide the statement of our regularity result in the context of problem (1.6).

Theorem 1.2. Given any $g \in L_p[0,T;L_2(\Gamma)]$ for $1 \leq p \leq \infty$, there is a unique strong solution y to problem (1.6). Moreover,

$$|y|_{H_p^{1/2,1/4}(Q)} \leq c|g|_{L_p[0,T;L_2(\Gamma)]} \qquad \text{for } 1 < p < \infty$$

and

$$|y|_{H_p^{1/2-2\varepsilon,1/4-\varepsilon}(Q)} \leq c|g|_{L_p[0,T;L_2(\Gamma)]} \qquad \text{for } p = 1, \infty, \text{ and } \varepsilon > 0.$$

//

The assertions of Theorem 1.2 follow immediately from Theorem 2.2 proved in Chapter II.

1.4 Main Results for the Parabolic Problem (1.6)

In this section, we shall provide for orientation purposes only qualitative statements of the main results, while the precise formulation with all the involved assumptions is relegated to chapter V. Also, for the sake of concretness, we shall state the main results in the context of the parabolic problem (1.6).

The main results in this work are of two kinds: those relating to the discrete-time approximation and those relating to the fully discrete approximation.

Before stating our results, we shall need to introduce some preliminary material.

First, we define the abstract operator A by

$$D(A) = \{u: A(x,\partial)u \in L_2(\Omega),\ u|_\Gamma \equiv 0\}$$

and

$$Au = -A(x,\partial)u \qquad \text{for } u \in D(A),$$

where $A(x,\partial)$ is a uniformly strongly elliptic, second-order differential operator given by (1.7). It is well-known [6] that $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup $S(t)$.

We also define the so-called "Dirichlet" map from the boundary Γ into the interior of Ω by

$$Dg = v \quad \text{if and only if} \quad A(x,\partial)\, v = 0 \quad \text{and} \quad v|_\Gamma = g.$$

It is well-known [14] that

$$D \in \mathcal{L}(H^s(\Gamma) \to H^{s+1/2}(\Omega)) \qquad \text{for all real } s.$$

The time-discrete approximation y_k is based on a backward Euler difference scheme and in the case of problem (1.6) has the formula (as we shall prove in Chapter IV)

$$(1.14)\quad \begin{cases} y_k(nk) = k \sum_{j=1}^{n} AE_k^{n+1-j} D(P_k g)(jk) & n = 1,\dots,N \\ y_k(t) = y_k(nk) & \text{for } (n-1)k < t \leq nk \end{cases}$$

where $E_k = \frac{1}{k} R(\frac{1}{k}, -A)$, P_k is an orthogonal projection on $L_p[0,T;L_2(\Omega)]$ or $L_p[0,T;L_2(\Gamma)]$ (we shall define P_k precisely in Chapter IV), and g is the boundary input. Here and thereafter $R(\lambda,-A)$ stands for the resolvent of A. We point out that AE_k is well defined since $AR(\lambda,-A) \in \mathcal{L}(L_2(\Omega) \to L_2(\Omega))$ (see section 4.3 for a more precise and more general statement, as well as the necessary justification).

The fully discrete approximation y_{hk} is based on the same backward Euler difference scheme, with the difference that the operator $\frac{1}{k} R(\frac{1}{k}, -A)$ is changed to $E_{hk} = \frac{1}{k} R(\frac{1}{k}, -A_h)$, where h is the parameter of space discretization, and where A_h is, for example, a Galerkin approximation of the operator A. A_h is defined on a family of finite-dimensional subspaces V_h, $0 < h < 1$, with suitably chosen approximation properties among which in particular the property that $D^*A^*v_h \in L_2(\Gamma)$ $\forall v_h \in V_h$ (we shall see later that this is equivalent to the property that $\frac{\partial}{\partial\nu} v_h \in L_2(\Gamma)$ $\forall v_h \in V_h$, which is a natural assumption for splines). The projection operator $P_h: L_2(\Omega) \to V_h$ is also needed, where P_h is orthogonal on $L_2(\Omega)$.

The approximation y_{hk} then has the formula

$$(1.15)\quad \begin{cases} y_{hk}(nk) = k \sum_{j=1}^{n} E_{hk}^{n+1-j} P_h(AD(P_k g)(jk)), & n=1,\ldots,N \\ y_{hk}(t) = y_{hk}(nk) & \text{for } (n-1)k < t \leq nk. \end{cases}$$

Since $D(AD) = \emptyset$ (the domain of A only contains functions which vanish on the boundary), the term $P_h(AD(P_k g)(jk))$ requires some explanation. In fact, since P_h is an orthogonal projection on $L_2(\Omega)$, we have

$$(P_h(AD(P_k g)(t)),v_h)_{L_2(\Omega)} = (AD(P_k g)(t),v_h)_{L_2(\Omega)} \qquad \forall v_h \in V_h.$$

Then

$$(1.16)\quad (AD(P_k g)(t),v_h)_{L_2(\Omega)} = \langle (P_k g)(t), D^* A^* v_h \rangle_{L_2(\Gamma)} \qquad \forall v_h \in V_h.$$

By virtue of the assumption $D^*A^*v_h \in L_2(\Gamma)$, the right-hand side of (1.16) is well-defined. This shows that $P_h(AD(P_k g),(t))$ is also well-defined.

Remark. We shall see in Chapter V that problem (1.15) is in fact a sequence of linear algebraic system of equations which can be solved numerically by using standard subroutines.

//

We are now in a position to state our main results, and we start with those directly related to the discrete-time approximation scheme.

Lemma 1.3. (See Lemma 4.1 in Chapter IV for a more general and precise statement).

(a) $$\|(E_k^n - S(nk))x\|_{L_2(\Omega)} \le \frac{c}{n} \|x\|_{L_2(\Omega)} \qquad n \ge 1$$

(b) $$\|A(E_k^n - S(nk))x\|_{L_2(\Omega)} \le \frac{c}{nk} \|x\|_{L_2(\Omega)} \qquad n \ge 1$$

//

In Huang-Thomée [9], this result is proved in the special case when a(u,v) is a V-elliptic bilinear symmetric form, where a(u,v) is the bilinear form associated to the operator A, and defined on a Hilbert space H with V a subspace of H. Using techniques different than those of Huang-Thomée [9], we shall prove Lemma 4.1 in a more general setting without assuming that A is selfadjoint. Lemma 4.1 is a key result in the proof of Theorem 4.2.

Theorem 1.4. (See Theorem 4.2 for a precise statement). Let y be the solution of problem (1.6) and let y_k be the solution of its discrete-time approximation as given in (1.14). Then

$$\|y - y_k\|_{L_p[0,T;L_2(\Omega)]} \le c k^{1/4 - \varepsilon} \|g\|_{L_p[0,T;L_2(\Gamma)]}$$

for $\varepsilon > 0$ given, and $1 \le p \le \infty$.

//

One can note that for $1 < p < \infty$, there is a loss of ε in the rate of convergence for $\|y - y_k\|$ as compared with the optimal regularity of the solution y (see section 1.3). We shall see that this loss is partially recovered in the fully discrete estimate (see Theorem 5.3).

In fact, for the fully discrete approximation, we have the following main result:

Theorem 1.5. (See Theorem 5.3 for the precise statement). Let y be the solution of problem (1.6) and let y_{hk} be the solution of its fully discrete approximation as given in (1.15). Then

$$|y-y_{hk}|_{L_p[0,T;L_2(\Omega)]} \le c(k^{1/4}\ln\frac{1}{k} + h^{1/2}\ln\frac{1}{h})|g|_{L_p[0,T;L_2(\Gamma)]}, \quad 1\le p\le\infty.$$

The estimates are subject to the restriction $k \le ch^2$ for some $c > 0$.

//

This restriction can be removed by imposing stronger approximation properties on the subspaces V_h. The removal of this restriction also carries the price of a slight loss in the rates of convergence for k and h. In fact, we have the following theorem.

Theorem 1.6. (See Theorem 5.4 for the precise statement). As in Theorem 5.1, let y be the solution to problem (1.6) and let y_{hk} be its fully discrete approximation as given in (1.15). With the additional assumption that $V_h \subseteq H^2(\Omega)$, we have

$$|y-y_{hk}|_{L_p[0,T;L_2(\Omega)]} \le c(k^{1/4-\varepsilon} + h^{1/2-2\varepsilon})|g|_{L_p[0,T;L_2(\Gamma)]}, \quad 1\le p\le\infty$$

without any restriction on k and h.

//

Remark. The spaces V_h have the property of best approximation, which means roughly speaking that the order of approximation of a given element v by elements of V_h is equal to the number of derivatives of v. The regularity of the solution y is $H_p^{1/2-2\varepsilon,\,1/4-\varepsilon}(Q)$ for $p = 1, \infty$ and $H_p^{1/2,\,1/4}(Q)$ for $1 < p < \infty$. In view of the best approximation property of V_h, the result of Theorem 5.3 is optimal for $p = 1, \infty$, and almost optimal (modulo $\ln \frac{1}{h}$ and $\ln \frac{1}{k}$) for $1 < p < \infty$. Similarly, the result of Theorem 5.4 is almost optimal (modulo ε) for $1 < p < \infty$ and essentially optimal for $p = 1, \infty$.

//

1.5 Survey of the Literature

In this section we shall collect the results of the available literature which are relevant to our work and we shall compare our results to those found there.

In our work we consider discrete Euler-Galerkin approximations of linear parabolic boundary-value problems with nonhomogeneous and nonsmooth Dirichlet boundary conditions. There is a wide range of problems, such as elliptic and parabolic problems, time-dependent problems, linear and non-linear problems, problems with Dirichlet, Neumann or mixed boundary conditions, continuous-time, discrete-time and fully discrete approximation schemes, taken in several combinations, to which certain common techniques are applicable.

Two basic approaches applicable in the situations described above are

(a) energy methods using variational arguments based on H^1-coercivity of the bilinear form associated with the operator $A(x,\partial)$

(b) a semigroup approach based on the analyticity of the semigroup generated by the operator $A(x,\partial)$ acting on some space with suitable boundary conditions.

The energy methods have a wide area of applicability, including for example elliptic and parabolic nonlinear problems. By the nature of their arguments, as seen, for example, in Douglas-Dupont [5], however, they are suited for treating only problems with high regularity of the input data.

The recently developed semigroup approach, applied to linear parabolic and hyperbolic problems (one can also treat some nonlinear problems by considering nonlinear terms as perturbations), permits the treatment of input data with significantly lower regularity requirements. This is especially significant in the area of control theory where control functions are in $L_p[0,T;L_2(\Gamma)]$ as a consequence of the natural physical assumptions.

Below we shall collect and review some results on approximation available in the literature and we shall attempt to relate these results to the present work.

In Douglas-Dupont [5], a number of problems are treated. The starting point is the study of a nonlinear parabolic equation with homogeneous Dirichlet boundary conditions. Galerkin-type approximation schemes are then developed for both the continuous case and the discrete case. Extensions to more general second order parabolic equations are considered, such as the time-dependent case, and nonhomogeneous boundary conditions. The points relevant to our work are that the second-order differential operator A is assumed to be selfadjoint, and that the methods used are variational in nature, based on H^1-coercivity of the bilinear form associated to A. Also, the convergence estimates involve H^1-norms of both the solution and its approximations. The solution is assumed to have all of its derivatives, with respect to the space variables and the time variable, in $L_2([0,T]x\Omega)$. This implies for example that the boundary data must be at least in $H^{1/2}(\Gamma)$ for the case of Dirichlet boundary conditions. As we have pointed out earlier, these methods are not applicable to the case of boundary data in $L_p[0,T;L_2(\Gamma)]$.

In the paper by Bramble-Schatz-Thomée-Wahlbin [4], semidiscrete Galerkin-type approximations for parabolic equations with homogeneous Dirichlet boundary conditions are considered. As in Douglas-Dupont, the operator A is assumed to be selfadjoint. L_2-estimates for the semidiscrete approximation, including estimates for L_2-initial data, are derived by using both energy inequalities and spectral representation of the solution.

As a digression, we point out that no convergence results are known as yet for the approximation of an elliptic equation with L_2-boundary data.

The extension of the results and techniques of this paper [4] to nonhomogeneous Dirichlet boundary conditions would again require that the boundary data be in $H^{1/2}(\Gamma)$.

The paper by Huang-Thomée [9] studies the backward Euler method applied to a general homogeneous parabolic initial-value problem in weak form.

Their main result states that

$$|U^n - u(nk)| \le \frac{c}{n} |u_o| \qquad \text{for } n = 1,\dots,N$$

where k is the parameter of discretization, N, u_o are given, and u(nk) is the actual value of the solution at time t = nk with U^n its approximate value. As we shall see, the statement of our preliminary Lemma 4.1 is essentially the same, except for the assumptions made. The authors assume the bilinear form to be time-dependent and V-elliptic; in our paper, although time-dependent problems are not considered, our assumption that -A generates an analytic semigroup generalizes the ellipticity condition. As a result, the techniques employed for the proof of our Lemma 4.1, based on an integral representation of the semigroup involving the resolvent, are very different from those used in Huang-Thomée [9].

The dissertation by Winther [21] treats the same type of problem as we do, i.e., the fully discrete approximation of a parabolic equation with nonhomogeneous boundary data. The one major difference is that he considers Neumann boundary conditions, as opposed to Dirichlet boundary conditions in our case. The significance of this difference can be seen by realizing that a standard variational approach requires the boundary term to be in $H^{-1/2}(\Gamma)$ for the Neumann problem versus $H^{1/2}(\Gamma)$ for the Dirichlet problem. Thus L_2 Neumann boundary conditions (which are the ones treated in Winther) are tractable with variational arguments. Winther also makes use of spectral representation of the solution, which requires the operator A to be selfadjoint.

Now we turn to results and works in which parabolic problems are treated with a semigroup approach.

In Lasiecka [10], a general and unified treatment of the regularity of solutions of parabolic problems for a large variety of nonhomogeneous boundary conditions is provided. The treatment is abstract in nature and the only assumption made on the operator A is that -A be the generator of an analytic semigroup. The ensuing theory which is developed forms the basis for a number of results in the area of approximation of singular integrals.

In fact, semidiscrete Galerkin approximations for parabolic problems with homogeneous boundary conditions have been considered in Lasiecka [11]. Although this particular paper is not directly relevant to our work (as it treats only homogeneous boundary conditions), it establishes several results which, like the uniform analyticity of the approximating semigroup, are crucially used to study nonhomogeneous boundary-value problems.

As a matter of fact, in Lasiecka [12], semidiscrete Galerkin approximations of parabolic problems with nonhomogeneous boundary data are studied. The distinctive feature of this paper is that $L_p[0,T;L_2(\Gamma)]$ (resp. $L_p[0,T;H^{-1}(\Gamma)]$) boundary conditions are treated in the Dirichlet case (resp. Neumann case). Using duality and interpolation theory, the optimal convergence estimates are derived for semidiscrete Galerkin approximations.

These results [12] were recently used by Mackenroth [15] to derive the convergence of a numerical algorithm for the approximation of abstract Riccati equations arising in the area of boundary control problems. These results [12] will also be used in our work to prove the stability of a fully discrete numerical algorithm which approximates the solution of problem (1.6).

To summarize, we have examined two different approaches for approximating parabolic boundary-value problems. First, the standard variational approach based on H^1-coercivity of the associated bilinear form is characterized by high rates of convergence, but on the other hand requires a high degree of smoothness of the boundary data, and is limited to differential operators satisfying some ellipticity condition.

The other semigroup approach is recent and has been developed to treat problems where the boundary data is "rough" or of low regularity. Also, the arguments are more general in nature from the point of view of the differential operator since one only assumes that $-A$ be the generator of an analytic semigroup.

Our work focused on "nonsmooth" boundary data is a continuation and extension of the latter semigroup approach [10]. The main results contained in Theorems 5.1, 5.2, 5.3, 5.4 on rates of convergence of the approximation to the "nonsmooth" abstract equation (1.11) as well as the "nonsmooth" parabolic equation (1.6) are new to the author's knowledge.

To obtain these results we need to develop new techniques based on the time and space approximation of singular integrals.

1.6 Outline of the Work

In the next chapter (chapter II) we first represent the solution of our abstract problem (1.11) by a semigroup formula.

We then recover the parabolic problem (1.6) from the abstract approach. At the end of chapter II regularity results for the solution to (1.6) are proved (Theorem 2.3).

In chapter III, we first provide a general treatment for the "space" discretization of the abstract problem (1.11). We then provide basic estimates for the error arising from the space approximation of the solution to (1.11) (see Theorem 3.1). Later we specialize these results to the canonical parabolic problem (1.6). For the (space) discrete approximation $u_h(t)$ defined by (3.25) we derive the basic error estimates in Theorem 3.3.

Chapter IV deals with discretization in time. There again, we first formulate the problem in an abstract setting. Theorem 4.2 states an abstract convergence result and later in section 4.5 we specialize this result to the parabolic case. The errors for the time-discrete approximation in the parabolic case are estimated in Theorem 4.4.

In chapter V we treat the full time and space discretization. Error estimates for the full discretization of the abstract problem (1.11) are given in Theorems 5.1 and 5.2. By specializing these abstract results to the parabolic case we then derive the main results on rates of convergence of the fully discrete approximation to the parabolic problem (1.6), stated in Theorems 5.3 and 5.4.

Some of the more lengthy and tedious proofs of results stated in this work are relegated to Chapter VI. Thus chapter VI consists of the proofs of Theorems 3.1, 4.2, 5.1, 5.2 as well as Lemma 4.1.

Finally, in chapter VII we present numerical results from a computer program which implements the full discretization scheme described in section 5.2 in the case of the canonical case of the heat equation.

CHAPTER II
SEMIGROUP REPRESENTATION FOR PARABOLIC BOUNDARY-VALUE PROBLEMS

2.1 Opening Remarks

As we have seen earlier, the treatment of boundary data in $L_p[0,T;L_2(\Gamma)]$ is not covered by standard methods and requires a new approach. Such a new approach is based on semigroup representation of the solution, and on the theory of singular integrals. This approach has been used in the context of abstract parabolic problems [3, 10] and in the context of semidiscrete Galerkin approximations [11, 12]. Our treatment of fully discrete Galerkin approximations uses some of the key ideas presented in Lasiecka [11, 12].

The basic stragegy is to reformulate problem (1.6) in the context of the abstract model (1.11). The solution of this abstract model has a natural semigroup formulation (see (1.12)). As we shall see later, the continuous-time Galerkin approximation of this abstract problem also has a natural semigroup formulation, the properties of which closely parallel those of the original formulation.

It is thus crucial to understand well the properties of the original problem in order to be able to derive the desired estimates for the continuous-time approximation. A case in point is that of the uniform analyticity of the approximating semigroup, which reflects the analyticity of the original semigroup.

This result, seemingly expected and straightforward, is on the contrary highly nontrivial and has been proved in papers by Huang and Thomée [8], Luskin and Rannacher [14], and Sammon [19] for the selfadjoint case and rather recently in Lasiecka [11] for the general case.

A similar statement can be made in the case of the discrete-time approximation. In fact, the operator E_k^n (see section (1.5)) approximates the continuous semigroup $S(t)$ for $t = nk$, and when considered as a discrete semigroup displays properties analogous to the analyticity of $S(t)$. These properties are essential to the proof of Lemma 4.1 (which states the approximation properties of E_k^n) which in turn is central in the derivation of the estimates for the discrete-time approximation.

For simplicity and convenience but without loss of generality we will concentrate on problem (1.13) with $f \equiv 0$ and $u_o \equiv 0$; i.e., we shall study the problem

$$(2.1)\quad \begin{cases} u_t = A(x,\partial)u & \text{on } Q \\ u(0,x) = 0 & \text{on } \Omega \\ u|_\Gamma = g & \text{on } \Sigma \end{cases}$$

where $A(x,\partial)u = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}u) + \sum_{i=1}^{n} b_i(x)D_{x_i}u + c(x)u$

$$= \operatorname{div}(M(x)\nabla u) + b(x)\cdot\nabla u + c(x)u$$

is a uniformly strongly elliptic, second order differential operator, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} \xi_i^2 \text{ for some } \gamma > 0, \text{ with } a_{ij}, b_i, c \in C^{\infty}(\Omega).$$

Also, as we have pointed out in section 1.2, we do not assume the matrix M(x) to be symmetric.

2.2 Semigroup Formula for the Solution of an Abstract Boundary-Value Problem

In this section we introduce an abstract framework into which problem (2.1) can be incorporated. To this end we shall introduce certain abstract operators.

Let A be a densely defined closed linear operator on a Hilbert space H ($A: D(A) \subseteq H \to H$). We assume that -A is the infinitesimal generator of a strongly continuous analytic semigroup S(t). It is well-known [18] that equivalently,

$$(2.2) \qquad \rho(-A) \supseteq \Sigma = \{\lambda: \operatorname{Re}\lambda > a - b|\operatorname{Im}\lambda|\} \qquad \text{for some } a,b > 0$$

and

$$(2.3) \qquad |R(\lambda,-A)|_{H\to H} \leq \frac{c}{1+|\lambda|} \qquad \forall\lambda \in \Sigma.$$

Equivalently to (2.3),

$$(2.4) \qquad |A^n S(t)|_{H\to H} \leq \frac{c}{t^n} e^{at} \quad , t > 0, n \geq 0.$$

We will assume without loss of generality that $a < 0$. Then $0 \in \rho(-A)$, which ensures that the fractional powers $A^{\alpha}, A^{*\alpha}$, $0 < \alpha < 1$ are well-defined [18]. In fact, if $a > 0$ it is enough to translate the generator to the left; i.e., we can take $\overline{A} = A + \overline{\lambda} I$ with $\overline{\lambda} > a$, and so

$$R(\lambda,-\overline{A}) = (\lambda+\overline{A})^{-1} = (\lambda+\overline{\lambda}+A)^{-1} = R(\lambda+\overline{\lambda},-A)$$

which shows that $\rho(-\overline{A}) \supseteq \Sigma$ with $a < 0$.

Also, the notation $D(A^{*-\alpha})$ which will be used several times in the sequel is defined by $D(A^{*-\alpha}) = D(A^{\alpha})'$, $0 < \alpha < 1$.

We now let $B: U \to D(A^*)'$ be a linear operator from a Hilbert space U into $D(A^*)'$, with the condition

$$(2.5) \qquad A^{-1}B \in \mathscr{L}(U \to H).$$

We consider the following abstract model:

$$(2.6) \qquad \begin{cases} u_t(t) = -Au(t) + Bf(t) & \text{on } D(A^*)' \\ u(0) = u_0 \end{cases}$$

where u_0 is given in H and $f \in L_p[0,T;U]$, $1 \leq p \leq \infty$.

Remark. Since $D(B)$ considered on U may be empty (this will be the case for our application), problem (2.6) should be understood in the sense of $D(A^*)'$-topology, i.e., as

$$(2.7)\quad \begin{cases} (u_t(t),v(t))_H = -(u(t),A^*v(t))_H + (f(t),B^*v(t))_U \\ \\ u(0) = u_0. \qquad\qquad \forall v(t) \in D(A^*), t \in (0,T] \end{cases}$$

//

The analyticity of the semigroup S(t) implies (as we shall see) the existence of a solution u(t) defined almost everywhere in the interval [0,T] for any given $f \in L_p[0,T;U]$.

The input-solution operator T for the problem (2.6) formally satisfies the semigroup formula

$$(2.8)\qquad u(t) = Tf(t) = S(t)u_0 + \int_0^t S(t-\tau)Bf(\tau)d\tau.$$

Remark. The kernel $S(t-\tau)B$ of the integral in (2.8) has the property that

$$|S(t-\tau)B|_{U\to H} = |AS(t-\tau)A^{-1}B|_{U\to H} \le c|AS(t-\tau)|_{H\to H}$$

by (2.5).

Then by analyticity of the semigroup S(t),

$$|AS(t-\tau)|_{H\to H} \le c \cdot \frac{1}{t-\tau},$$

$1/(t-\tau)$ being the kernel of the Hilbert transform. Therefore the integral in (2.8) should be considered as a singular integral.

//

At this point we again assume for simplicity, but without loss of generality, that $u_0 \equiv 0$.

We shall now see that the formula (2.8) defines almost everywhere in t an element in H for $f \in L_p[0,T;U]$. If we assume as in (2.5) but more generally that

$$(2.9) \qquad A^{-q}B \in \mathcal{L}(U \to H) \qquad\qquad \text{for some } 0 \le q \le 1,$$

T can be rewritten as

$$(2.10) \qquad Tf(t) = A^q \int_0^t S(t-\tau)A^{-q}Bf(\tau)d\tau.$$

The following regularity result for T is proved in Lasiecka [12], where the theory of singular integrals [20] plays a central role in the arguments.

Theorem 2.1 [12]. Let $0 \le q \le 1$ and assume condition (2.9). Then

(a) $\qquad T \in \mathcal{L}([L_p[0,T;U] \to L_p[0,T;D(A^{*1-q})]) \qquad$ for $1 < p < \infty$

(b) $\qquad T \in \mathcal{L}(L_p[0,T;U] \to L_p[0,T;D(A^{*1-q-\varepsilon})]) \qquad$ for $p = 1, \infty, \varepsilon>0$.

//

We shall also need

Theorem 2.2. Let $0 \le q \le 1$ and assume condition (2.9). Then for T satisfying formula (2.10), we have

(a) $T \in \mathcal{L}(L_p[0,T;U] \to H_p^{1-q}[0,T;H])$ for $1 < p < \infty$

(b) $T \in \mathcal{L}(L_p[0,T;U] \to H_p^{1-q-\varepsilon}[0,T;H])$ for $p = 1, \infty, \varepsilon>0$.

//

Proof. We have

$$\frac{dT}{dt}(t) = Bf(t) + A \int_0^t S(t-\tau)Bf(\tau)d\tau.$$

Then

$$A^{-q} \frac{dT}{dt}(t) = \hat{f}(t) + A \int_0^t S(t-\tau)\hat{f}(\tau)d\tau$$

where $\hat{f}(t) = A^{-q}Bf(t)$. By assumption on B and f, we know that $\hat{f} \in L_p[0,T;H]$. Let

$$\hat{T}\hat{f}(t) = A \int_0^t S(t-\tau)\hat{f}(\tau)d\tau.$$

In the case $1 < p < \infty$, by applying part (a) of Theorem 2.1 to $\hat{T}$ with $B = I$ and $q = 1$, we obtain

$$(2.11) \qquad A^{-q} \frac{dT}{dt} \in \mathcal{L}(L_p[0,T;U] \to L_p[0,T;H]),\ 1 < p < \infty.$$

Similarly, in the case $p = 1, \infty$, by applying part (b) of Theorem 2.1 to $\hat{T}$ with $B = I$ and $q = 1-\varepsilon$, we obtain

$$(2.12) \qquad A^{-q-\varepsilon} \frac{dT}{dt} \in \mathcal{L}(L_p[0,T;U] \to L_p[0,T;H]),\ p = 1, \infty.$$

We now apply the intermediate derivatives theorem [13, Vol. I, p. 15] whose original formulation in the setting of L_2 spaces can be extended to L_p spaces, to (2.11) and (2.12), which yields (a) and (b) as claimed.

//

These regularity results will be used to derive corresponding regularity results for our canonical problem (2.1).

2.3 Application to a Canonical Example of a Parabolic Boundary-Value Problem

The goal in this section is to derive a semigroup formula for the solution of problem (2.1). This will be achieved by expressing it in the context of section 2.2. Thus we set $H = L_2(\Omega)$, $U = L_2(\Gamma)$. We also need to introduce a few operators.

Let $D: L_2(\Gamma) \to L_2(\Omega)$ be the so-called "Dirichlet" map, which is a natural extension into the interior of Ω of a function defined on the boundary Γ. By definition,

$$Dg = v \qquad \text{if and only if} \qquad \begin{array}{l} A(x,\partial)\, v = 0 \\ \text{and } v|_\Gamma = g. \end{array}$$

It is well-known [16] that

$$(2.13) \quad D \in \mathcal{L}(H^s(\Gamma) \to H^{s+1/2}(\Omega)) \qquad \text{for all real } s.$$

We also define the operator A: $L_2(\Omega) \to L_2(\Omega)$ by

$$D(A) = \{f: A(x,\partial)f \in L_2(\Omega),\ f|_\Gamma = 0\}$$

and

$$Af = -A(x,\partial)f \qquad \text{for } f \in D(A).$$

It is well-known [6] that -A is the infinitesimal generator of a strongly continuous analytic semigroup S(t) on $L_2(\Omega)$.

We begin the derivation of the desired semigroup formula with the following standard argument, in which we need to assume that $g \in H_p^{2,1}(\Sigma)$ (later we shall extend the formula to all of $L_p[0,T;L_2(\Gamma)]$ by density).

Let $\tilde{y} = y - Dg$. Then, similarly as in section 1.2,

$$y_t = \tilde{y}_t + (Dg)_t \text{ and } \qquad A(x,\partial)y = A(x,\partial)\tilde{y} + A(x,\partial)Dg$$
$$= A(x,\partial)\tilde{y}$$

since by definition $A(x,\partial)Dg = 0$.

Problem (2.1) then becomes:

$$(2.14) \qquad \begin{cases} \tilde{y}_t = A(x,\partial)\tilde{y} - (Dg)_t & \text{on } Q \\ \tilde{y}(0,x) = -Dg(0,x) & \text{on } \Omega \\ \tilde{y}|_\Gamma = 0 & \text{on } \Sigma. \end{cases}$$

Since $\tilde{y}$ satisfies zero Dirichlet boundary conditions, problem (2.14) can be put into the form of an abstract boundary-value problem by defining the following "abstract" variables:

$$y(t) = y(t,\cdot) \quad \in L_2(\Omega),$$

$$\tilde{y}(t) = \tilde{y}(t,\cdot) \quad \in L_2(\Omega),$$

$$g(t) = g(t,\cdot) \quad \in L_2(\Gamma) \qquad \text{for } t > 0.$$

Then the abstract problem

$$(2.15) \qquad \begin{cases} \tilde{y}_t = -A\tilde{y} - D(g_t) & \text{on } Q \\ \tilde{y}(0) = -Dg(0) & \text{on } \Omega \end{cases}$$

is a special case of problem (2.6) with $B = -D$ and $u_o = -Dg(0)$. We note that $D(g_t)$ is well-defined since g is assumed to be in $H_p^{2,1}(\Sigma)$. We also note that the boundary conditions are explicit in (2.14) whereas they are implicit in (2.15). ($D(A)$ only contains functions which vanish on the boundary).

The variation of parameter formula gives the following representation for the solution of (2.15):

$$(2.16) \qquad \tilde{y}(t) = -S(t)Dg(0) - \int_0^t S(t-\tau)(Dg)_t(\tau)d\tau.$$

Our objective is to obtain a corresponding formula for u(t). Integrating by parts, which is justified since $g \in H_p^{2,1}(\Sigma)$, gives

$$\int_0^t S(t-\tau)(Dg)_t(\tau)d\tau = [S(t-\tau)Dg(\tau)]_{\tau=0}^{\tau=t} - \int_0^t \frac{d}{d\tau} S(t-\tau)Dg(\tau)d\tau$$

$$= Dg(t) - S(t)Dg(0) - \int_0^t AS(t-\tau)Dg(\tau)d\tau.$$

Since $y(t) = \tilde{y}(t) + Dg(t)$, we can see that

$$(2.17)\qquad y(t) = A\int_0^t S(t-\tau)Dg(\tau)d\tau$$

is the desired semigroup formula for the solution of problem (2.1). Since

$$(2.18)\qquad A^{\alpha}S(t)x = S(t)A^{\alpha}x \qquad \text{for all real } \alpha, \text{ and } x \in D(A^{\alpha}),$$

we can formally rewrite (2.17) by using (2.18), as

$$(2.19)\qquad y(t) = \int_0^t S(t-\tau)ADg(\tau)d\tau.$$

Now we can see that y is the solution of the abstract problem

$$(2.20)\qquad \begin{cases} y_t = -Ay + ADg & \text{on } D(A^*)' \\ y(0) = 0 \end{cases}$$

of which the canonical example is of course (2.1). Since $Dg|_{\Gamma} = g \neq 0$ in general, it follows that $D(AD) = \emptyset$. It is therefore important to note that both (2.19) and (2.20) should be understood (as we have pointed out in the remark of section 2.2) in the sense of $D(A^*)'$ - topology. A proper way of stating (2.20) would be for example

$$(2.21)\qquad \begin{cases} (u_t(t),v)_{\Omega} = (-Au(t),v)_{\Omega} + (ADg(t),v) \\ \qquad\qquad = -(u(t),A^*v)_{\Omega} + \langle g(t),D^*A^*v\rangle_{\Gamma} \text{ for } v \in D(A^*) \\ u(0) = 0. \end{cases}$$

The semigroup formula (2.17) is defined so far for $g \in H_p^{2,1}(\Sigma)$, but we shall show in the next section that it can be extended by density to $g \in L_p[0,T;L_2(\Gamma)]$.

2.4 Regularity of the Solution to (2.1)

Let $L\colon L_p[0,T;L_2(\Gamma)] \to L_p[0,T;L_2(\Omega)]$ be the input-solution operator for problem (2.1), i.e.,

$$(2.22) \qquad Lg(t) = A \int_0^t S(t-\tau)Dg(\tau)d\tau.$$

L is certainly well-defined on $H_p^{2,1}(\Sigma)$ as we have seen in section 2.2, so that $H_p^{2,1}(\Sigma) \subseteq D(L)$. Since $H_p^{2,1}(\Sigma)$ is dense in $L_p[0,T;L_2(\Gamma)]$ it follows that L is densely defined.

We shall now state and prove our regularity result for the solution of problem (2.1) given by the semigroup formula (2.22).

Theorem 2.3.

(a) $L \in \mathcal{L}(L_p[0,T;L_2(\Gamma)] \to H_p^{1/2,\,1/4}(Q))$ for $1 < p < \infty$

(b) $L \in \mathcal{L}(L_p[0,T;L_2(\Gamma)] \to H_p^{1/2-2\varepsilon,\,1/4-\varepsilon}(Q))$ for $p = 1, \infty$, $\varepsilon>0$.

//

Proof. Knowing that [7]

$$(2.23)\qquad D(A^{\alpha}) = H^{2\alpha}(\Omega) \qquad\qquad \text{for } 0 \le \alpha < 1/4 ,$$

we have by (2.13)

$$(2.24)\qquad D \in \mathcal{L}(L_2(\Gamma) \to D(A^{1/4-\varepsilon})) \qquad \text{for every } \varepsilon > 0.$$

This fact will be used repeatedly in the sequel.

We first prove the "easier" part (b).

The semigroup formula (2.22) can be rewritten equivalently as

$$(2.25)\qquad y(t) = Lg(t) = A^{3/4+\varepsilon}\int_0^t S(t-\tau)A^{-3/4-\varepsilon}(AD)g(\tau)d\tau \quad \text{for } \varepsilon>0.$$

Thus we are exactly in the situation described in (2.6), with $H = L_2(\Omega)$, $U = L_2(\Gamma)$, $B = AD$ (formally), and with $q = 3/4+\varepsilon$ in (2.9). With these identifications, it follows from part (b) of Theorem 2.1 and from (2.22) that

$$(2.26)\qquad L \in \mathcal{L}(L_p[0,T;L_2(\Gamma)] \to L_p[0,T;H_p^{1/2-2\varepsilon}(\Omega)]),\ \varepsilon > 0.$$

The time regularity of L follows from part (b) of Theorem 2.2 with the same identifications. Thus we have

$$(2.27)\qquad L \in \mathcal{L}(L_p[0,T;L_2(\Gamma)] \to H_p^{1/4-\varepsilon}[0,T;L_2(\Omega)]),\ \varepsilon > 0.$$

This completes the proof of (b), as

$$H_p^{1/2-2\varepsilon,\,1/4-\varepsilon}(Q) = L_p[0,T;H^{1/2-2\varepsilon}(\Omega)] \cap H_p^{1/4-\varepsilon}[0,T;L_2(\Omega)].$$

We now prove (a). This will be achieved by interpolating the two following results.

(a_1) $\quad L \in \mathcal{L}(\tilde{H}_p^{3/2,3/4}(\Sigma) \to H_p^{2,1}(Q)) \qquad 1 < p < \infty,$

(a_2) $\quad L \in \mathcal{L}(\tilde{H}_p^{-1/2,-1/4}(\Sigma) \to L_p[0,T;L_2(\Omega)]) \qquad 1 < p < \infty$

where $\tilde{H}_p^{r,s}(\Sigma) = \{f \in H_p^{r,s}(\Sigma)\colon \; f(0) = y_o|_\Gamma = 0\}$, i.e., the elements of $\tilde{H}_p^{r,s}(\Sigma)$ satisfy compatibility relations.

We point out that

$$(2.28) \qquad \tilde{H}_p^{r,s}(\Sigma) = H_p^{r,s}(\Sigma) \qquad \text{for } |r| \le 1/2, |s| \le 1/4, \quad 1 \le p \le \infty.$$

Thus, in the space of solutions $H_p^{1/2,1/4}(Q)$, the compatibility relations also disappear. This is due to the fact that $H_o^{1/2}(\Omega) = H^{1/2}(\Omega)$; i.e., the topology of $H_o^{1/2}(\Omega)$ is too weak to "recognize" boundary values.

First, we show (a_1). Because of the high regularity of the boundary input, we shall use a standard argument.

Let $\tilde{y} = y - \tilde{g}$, where $\tilde{g}$ extends g from Σ into the interior of Q, in such a way that the compatibility relations are preserved; i.e., $\tilde{g}(0) = y_o = 0$ on Ω. By inverse trace theory, we know that if $g \in H_p^{3/2,3/4}(\Sigma)$, then $\tilde{g} \in H_p^{2,1}(Q)$ [13, Vol. II, p.9]. Therefore, it is sufficient to establish the same regularity for $\tilde{u}$.

We recall from section 1.2 that $\tilde{y}$ satisifes the boundary-value problem

$$(2.29)\quad \begin{cases} \tilde{y}_t = -A\tilde{y} + f & \text{on } Q \\ \tilde{y}(0) = -\tilde{g}(0) & \text{on } \Omega \end{cases}$$

where $f(t) = A(x,\partial)\tilde{g}(t) - \tilde{g}_t(t)$. Since $\tilde{g} \in H_p^{2,1}(Q)$, it follows that $f \in L_p[0,T;L_2(\Omega)]$. Then the solution of (2.29) is given by the semigroup formula

$$\begin{aligned}\tilde{y}(t) &= -S(t)\tilde{g}(0) + \int_0^t S(t-\tau)f(\tau)d\tau \\ &= \int_0^t S(t-\tau)f(\tau)d\tau\end{aligned}$$

because of the compatibility relation $\tilde{g}(0) = 0$. Hence

$$A\tilde{y}(t) = A\int_0^t S(t-\tau)f(\tau)d\tau.$$

By application of Theorem 2.1 with $U = H = L_2(\Omega)$, $B = A$ and $q = 1$, we get

$$(2.30)\quad |\tilde{y}|_{L_p[0,T;D(A)]} = |A\tilde{y}|_{L_p[0,T;L_2(\Omega)]} \leq c|f|_{L_p[0,T;L_2(\Omega)]},\ 1<p<\infty.$$

Using the identification [7]

$$D(A) = D(A^*) = H^2(\Omega) \cap H_o^1(\Omega)$$

and

$$|x|_{H^{2\beta}(\Omega)} \leq c|x|_{D(A^\beta)} \qquad \text{for } x \in D(A^\beta),\ 0 \leq \beta \leq 1,$$

we can see that

$$|\tilde{y}|_{L_p[0,T;H^2(\Omega)]} \leq c|f|_{L_p[0,T;L_2(\Omega)]}, \quad 1 < p < \infty.$$

As for the time regularity of $\tilde{u}$, we have

$$\tilde{y}_t(t) = -A\tilde{y}(t) + f(t) = -A \int_0^t S(t-\tau)f(\tau)d\tau + f(t)$$

and so by (2.30) we obtain

$$|\tilde{u}_t|_{L_p[0,T;L_2(\Omega)]} \leq c|f|_{L_p[0,T;L_2(\Omega)]} \qquad 1 < p < \infty.$$

Therefore (a_1) is proved since $H_p^{2,1}(Q) = L_p[0,T;H^2(\Omega)] \cap H_p^1[0,T;L_2(\Omega)]$

We now need to prove

$$(a_2) \qquad L \in \mathscr{L}(H_p^{-1/2,-1/4}(\Sigma) \to L_p[0,T;L_2(\Omega)]) \qquad 1 < p < \infty.$$

Assuming the validity of (a_2), our result follows directly from the application of a general interpolation theorem [13, Vol. I, p. 27]. More precisely, the theorem concludes that

$$L \in \mathscr{L}([\tilde{H}_p^{3/2,3/4}(\Sigma), \tilde{H}_p^{-1/2,-1/4}(\Sigma)]_\theta \to [H_p^{2,1}(Q), H_p^{0,0}(Q)]_\theta)$$

for $0 \le \theta \le 1$.

Since we are concerned with $g \in L_p[0,T;L_2(\Gamma)] = H_p^{0,0}(\Sigma)$, we take $\theta = 3/4$. By our remark (2.28), we deduce that

$$L \in \mathcal{L}(L_p[0,T;L_2(\Gamma)] \to H_p^{1/2,1/4}(Q))$$

as desired.

As for the proof of (a_2) we shall use a simple but useful duality result. We state and prove this result in an abstract context as it will be used repeatedly throughout this work. First of all, we define the adjoint of an operator.

Definition. Let $B: U \to H$ be a densely defined linear operator, where U, H are arbitrary Hilbert spaces. Then the adjoint B^* of B is defined by $B^*v = z$, where

$$(2.31) \qquad (Bu,v)_H = (u,z)_U \qquad \text{for all } u \in D(B)$$

and $D(B^*)$ is the set of all z such that condition (2.29) is satisfied for some $z \in U$.

Now we have

Lemma 2.4. Let K be an operator such that

$$K \in \mathcal{L}(L_2[0,T;U] \to L_2[0,T;H]),$$

where U, H are arbitrary Hilbert spaces and $T > 0$ is fixed.

Then

$$K \in \mathcal{L}(L_p[0,T;U] \to L_p[0,T;H]) \text{ implies that}$$

$$K^* \in \mathcal{L}(L_q[0,T;H] \to L_q[0,T;U]),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < \infty$ and $q = \infty$ (resp. 1) for $p = 1$ (resp. ∞).

//

Proof. The result is immediate for $1 \leq p < \infty$ by duality and by definition of the adjoint. We now prove the case $p = \infty$.

First, by definition of K^*,

$$\int_0^t (Ku(t),v(t))_H dt = \int_0^t (u(t),K^*v(t))_U dt \qquad \forall u \in D(K),\ v \in D(K^*). \tag{2.32}$$

Let $v \in L_1[0,T;H]$, and let $\{v_n\}$ be a sequence in $L_2[0,T;H]$ such that $v_n \to v$ in $L_1[0,T;H]$. Then $K^*v_n \in L_2[0,T;U]$ for each n by Lemma 2.4 with $p = 2$. With $u \in L_\infty[0,T;U]$, consider

$$\int_0^t (Ku(t),v_n(t))_H dt = \int_0^t (u(t),K^*v_n(t))_U dt \tag{2.33}$$

Then by assumption

$$(2.34) \qquad \int_0^t (Ku(t),v_n(t))_H dt \leq c|u|_{L_\infty[0,T;U]} |v_n|_{L_1[0,T;H]}$$

$$\leq c|u|_{L_\infty[0,T;U]} |v|_{L_1[0,T;H]}$$

We now let $u(t) = \dfrac{K^*v_n(t)}{|K^*v_n(t)|_U}$ which is well-defined in $L_\infty[0,T;U]$ as

$K^*v_n(t) \in U$ almost everywhere in t. Hence by (2.33) and (2.34),

$$(2.35) \qquad \int_0^t |K^*v_n(t)|_U dt \leq c \cdot 1 \cdot |v_n|_{L_1[0,T;H]}$$

$$\leq c|v|_{L_1[0,T;H]}$$

as $|u(t)|_U = 1$ almost everywhere in t.

By passing to the limit in (2.35), we deduce that K^* can be extended to a bounded operator from $L_1[0,T;H]$ into $L_1[0,T;U]$.

//

Coming back to the proof of (a_2), it is therefore sufficient to show that

$$L^* \in \mathscr{L}(L_p[0,T;L_2(\Omega)] \to H_p^{1/2,\,1/4}(\Sigma)) \qquad \text{for } 1 < p < \infty.$$

We shall now derive the formula for $L^*v(t)$ in an abstract context. This abstract formulation will be extensively used in the subsequent chapters and its derivation is straightforward.

Let T be given by the formula (2.10), i.e.

$$(2.36) \quad Tu(t) = \int_0^t S(t-\tau)Bu(\tau)d\tau.$$

Then by definition of the adjoint T^*, we have

$$(Tu,v)_{L_2[0,T;H]} = \int_0^T (Tu(t),v(t))_H dt = \int_0^T (u(t),T^*v(t)_U dt$$

$$= (u,T^*v)_{L_2[0,T;U]}.$$

Also,

$$\int_0^T (Tu(t),v(t)_H dt = \int_0^T (\int_0^t S(t-\tau)Bu(\tau)d\tau, v(t))_H dt$$

$$= \int_0^T \int_0^t (S(t-\tau)Bu(\tau),v(t))_H d\tau dt.$$

By reversing the order of integration, we have

$$\int_0^T (Tu(t),v(t))_H dt = \int_0^T \int_\tau^T (u(\tau),B^*S^*(t-\tau)v(t))_U dt d\tau$$

$$= \int_0^T (u(\tau), \int_\tau^T B^*S^*(t-\tau)v(t)dt)_U d\tau$$

$$= \int_0^T (u(\tau),T^*v(\tau))_U d\tau.$$

Hence

$$(2.37) \quad T^*v(t) = B^* \int_t^T S^*(\tau-t)v(\tau)d\tau.$$

By taking $H = L_2(\Omega)$, $U = L_2(\Gamma)$ and $B = AD$ formally with $q = \frac{3}{4} + \varepsilon$, $\varepsilon > 0$ we deduce the formula for $L^*v(t)$, i.e,

$$L^*v(t) = D^*A^* \int_t^T S^*(\tau - t)v(\tau)d\tau \qquad \text{on } L_2(\Omega).$$

Now we define the auxiliary operator M^* by

$$M^*v(t) = \int_t^T S^*(\tau - t)v(\tau)d\tau.$$

Our strategy is to first establish a certain regularity for M^* and then to quote a standard result from trace theory to conclude the argument.

We recall that v is assumed to be in $L_p[0,T;L_2(\Omega)]$ for $1 < p < \infty$. By applying Theorem 2.1 with $U = H = L_2(\Omega)$ and $B = I$ with $q = 0$, we have

$$|M^*v|_{L_p[0,T;D(A^*)]} \leq c|v|_{L_p[0,T;L_2(\Omega)]}, \qquad 1 < p < \infty.$$

Using the identification [7]

$$D(A) = D(A^*) = H^2(\Omega) \cap H_o^1(\Omega)$$

and

$$|x|_{H^{2\beta}(\Omega)} \leq c|x|_{D(A^{*\beta})} \qquad \text{for } x \in D(A^{*\beta}),\ 0 \leq \beta \leq 1,$$

we can see that

$$|M^*v|_{L_p[0,T;H^2(\Omega)]} \leq c|v|_{L_p[0,T;L_2(\Omega)]}, \qquad 1 < p < \infty.$$

As for the time regularity of M^*v,

$$\frac{dM^*v}{dt}(t) = -v(t) + \int_t^T A^*S^*(\tau-t)v(\tau)d\tau.$$

By applying Theorem 2.1 again with $U = H = L_2(\Omega)$ and $B = A$ with $q = 1$, we get

$$\left|\frac{dM^*v}{dt}\right|_{L_p[0,T;L_2(\Omega)]} \leq c|v|_{L_p[0,T;L_2(\Omega)]}, \qquad 1 < p < \infty.$$

Thus we have shown that

$$(2.38) \quad M^* \in \mathcal{L}(L_p[0,T;L_2(\Omega)] \to H_p^{2,1}(Q)), \qquad 1 < p < \infty.$$

To finish the argument, we claim that

$$(2.39) \quad D^*A^*v = \frac{\partial v}{\partial \nu_A} \qquad \text{for } v \in D(D^*A^*),$$

where $\frac{\partial v}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x)n_i(x)\frac{\partial v}{\partial x_j}$ and $n_i(x)$ are the components of the outward unit vector normal to the boundary Γ at x. This can easily be seen by applying the second Green's formula to $A(x,\partial)$:

$$(A(x,\partial)u,v)_\Omega - (u,A^*(x,\partial)v)_\Omega = \langle\frac{\partial u}{\partial \nu_A},v\rangle_\Gamma - \langle u,\frac{\partial v}{\partial \nu_A}\rangle_\Gamma.$$

By choosing $u = Dg$ and $v \in D(A^*)$ (i.e. $v|_\Gamma = 0$), we obtain

$$-(Dg,\ A^*(x,\partial)v)_\Omega = -\langle g, D^*A^*v\rangle_\Gamma = -\ \langle g, \frac{\partial v}{\partial \nu_A}\rangle_\Gamma$$

for all $v \in D(D^*A^*)$, $g \in D(D^*A^*)'$.

The following theorem, which was originally formulated in the setting of L_2 spaces [13, Vol. II, p. 9], can be extended to L_p spaces for $1 < p < \infty$ and provides us with the regularity result we need.

Theorem [13]. The mapping from $H_p^{r,s}(Q)$ into $H_p^{\mu_j,\nu_j}(\Sigma)$ which to u associates $\frac{\partial^j u}{\partial \nu_A^j}$ is continuous, where

$$\frac{\mu_j}{r} = \frac{\nu_j}{s} = \frac{r-j-\frac{1}{2}}{r} \qquad \text{for } r > \tfrac{1}{2},\quad s \geq 0.$$

//

By taking $r = 2$, $s = 1$, $j = 1$, we obtain

$$\left|\frac{\partial v}{\partial \nu_A}\right|_{H_p^{1/2,\,1/4}(\Sigma)} \leq c\,|v|_{H_p^{2,1}(Q)},$$

and thus finally

$$\begin{aligned} |L^*v|_{H_p^{1/2,\,1/4}(\Sigma)} &= \left|\frac{\partial M^*v}{\partial \nu_A}\right|_{H_p^{1/2,\,1/4}(\Sigma)} \\ &\leq c\,|M^*v|_{H_p^{2,1}(Q)} \\ &\leq c\,|v|_{L_p[0,T;L_2(\Omega)]}. \end{aligned}$$

This concludes the proof of (a_2).

Remark. Theorem 1.2 is a direct consequence of Theorem 2.3 since y in Theorem 1.2 and L in Theorem 2.3 are related by

$$y(t) = Lg(t), \qquad t > 0.$$

//

CHAPTER III
SPACE DISCRETIZATION

3.1 Opening Remarks

In this chapter we shall introduce an approximation scheme which, given a family of finite-dimensional subspaces V_h and a family of finite-dimensional operators A_h, produces for any fixed value of t an element $u_h(t)$ in V_h which approximates at time t the solution u(t). This approximation scheme, which is generally called continuous-time approximation because the discretization does not affect the time variable, will be presented in an abstract context. An optimal abstract convergence result will then be given. Both the approximation scheme and the convergence result are found in Lasiecka [12].

The abstract scheme and its convergence estimates are then applied to the case of our canonical example (2.1). A family of standard approximating subspaces, as well as the standard Galerkin approximation A_h of the operator A, are shown to give rise to an approximation scheme with optimal convergence properties by virtue of the applicability of the abstract scheme to these choices.

We emphasize here that this applicability, based on the uniform analyticity of the semigroup generated by $-A_h$, is a nontrivial result which has been established only recently in the selfadjoint case [8, 14, 19], and in the general case [11]. This point will be further elaborated in sections 3.2 and 3.4.

3.2 Galerkin Approximation of an Abstract Boundary-Value Problem

In this section we shall provide a general framework for the approximation of abstract boundary-value problems.

Let h be the parameter of discretization, with $0 < h < 1$ and with its values approaching zero. Let $V_h \subseteq D(B^*)$ be a family of finite-dimensional subspaces of H and let P_h be the orthogonal projection of H onto V_h. We shall assume that the subspaces V_h enjoy the following approximation properties:

$$(3.1)\qquad |x-P_h x|_H \leq ch^{2\alpha}|x|_{D(A^\alpha)},\ 0 \leq \alpha \leq 1,$$

$$(3.2)\qquad |B^*(I-P_h)x|_U \leq ch^{2(1-\bar{q})}|x|_{D(A^*)},$$

$$(3.3)\qquad |B^*(I-P_h)x|_U \leq ch^{2\alpha}|x|_{D(A^{*\alpha+q})},\ 0 \leq \alpha \leq 1-q,$$

$$(3.4)\qquad |B^* v_h|_U \leq ch^{-2\bar{q}}\ |v_h|_H \quad \text{(inverse approximation property)},$$

for some $\bar{q} \leq q$.

Another inverse approximation property which is stronger than (3.4) is

$$(3.4') \qquad |v_h|_{D(A^{*\alpha})} \le ch^{-2\alpha}|v_h|_H, \qquad 0 \le \alpha \le 1.$$

We will need to assume (3.4') in the context of Theorem 5.2 rather than (3.4).

As a preview to section 3.4, we point out that the inverse assumption (3.4') is still reasonably general as it is satisfied for example by spaces of splines defined on uniform meshes.

The operator A will be approximated by a family of finite-dimensional operators $A_h: V_h \to V_h$ satisfying the following generic properties:

$$(3.5) \qquad |(A^{-1}-A_h^{-1}P_h)x|_H \le ch^2\|x|_H \qquad \text{(convergence)},$$

$$(3.6) \qquad |A_h^{\beta}S_h(t)|_{H\to H} \le \frac{c_\beta}{t^\beta}, \ 0 \le \beta \le 2 \quad \text{uniformly in } h,$$

where $S_h(t) = \exp(-A_h t)$. Property (3.6) is referred to as uniform analyticity of the semigroup $S_h(t)$.

We also assume that the properties (3.5) - (3.6) hold for A*.

Equivalently to (3.6), we have [11]

$$(3.7) \qquad \rho(-A_h) \quad \Sigma = \{\lambda: \mathrm{Re}\lambda > a - b|\mathrm{Im}\lambda|\} \quad \text{for some } a,\ b > 0$$

and

$$(3.8) \qquad |R(\lambda,-A_h)P_h|_{H\to H} \le \frac{c}{1+|\lambda|} \qquad \text{uniformly in } h.$$

(3.7) and (3.8) can be compared with (2.2) and (2.3) in section 2.2.

Remark. If A is coercive in the norm of $D(A^{1/2})$ (which is the case for the canonical example 2.1), then the Galerkin approximation A_h defined by

$$(A_h x_h, y_h)_H = (A x_h, y_h)_H \in \forall x_h, y_h \in V_h \subseteq D(A^{1/2})$$

satisfies (3.5) and (3.6) (and consequently (3.7) and (3.8)). Although this result is to be expected, it has been shown only recently [11] and its proof is technical.

//

Let u(t) be the solution of the abstract boundary-value problem (2.6). We shall consider the following approximating scheme:

find $u_h(t)$ V_h such that

$$(3.9) \quad \begin{cases} (\dot{u}_h(t), v_h)_H = -(A_h u_h(t), v_h)_H + (g(t), B^* v_h)_U & \forall v_h \in V_h \\ u_h(0) = 0 \end{cases}$$

Problem (3.9) can be considered as a semidiscrete approximation of problem (2.6). To see this, we define the operator T_h: $L_2[0,T;U] \to L_2[0,T;V_h]$ corresponding to the input-solution operator T for the problem (2.6), by

$$(3.10) \quad T_h g(t) = u_h(t) = \int_0^t S_h(t-\tau) P_h (Bg)(\tau) d\tau,$$

which is the (formal) semigroup formula for the solution of (3.9). Thus (3.10) is an approximation of (2.10).

Remark. The operator T_h is well defined for any input $g \in L_p[0,T;U]$, $1 \leq p \leq \infty$. This can be seen by using the properties of P_h and the subspaces V_h. In fact, for $g \in U$, and from the orthogonality of P_h, we have

$$(P_h(Bg),v)_H = (P_h(Bg),P_h v)_H = (Bg,P_h v)_H \quad \forall v \in H.$$

Now let $v_h \in V_h \subseteq D(B^*)$. Then

$$(Bg,v_h)_H = (g,B^* v_h)_U \quad \forall g \in U,\ \forall v_h \in V_h.$$

Thus we have shown that $P_h(Bg)$ is well-defined in H for $g \in U$, which in turn shows that T_h is well-defined.

//

3.3 Estimation of the Error

In this section we shall provide results on the rate of convergence of the solution of (3.9) to the solution of (2.6).

Let $e(t) = u(t) - u_h(t) = (T-T_h)g(t)$, where $u(t)$, $u_h(t)$ are the solutions to (2.6) and (3.9) respectively. As in the proof of Theorem 2.3 (regularity), we shall use Lemma 2.4 (duality result) by establishing error estimates for $(T^*-T_h^*)$. By virtue of Lemma 2.4, the same estimates will then hold for $(T-T_h)$. Thus in order to establish the rates of convergence for $e(t)$ it is enough to estimate $T^*-T_h^*$.

We shall now derive the formula for the adjoint T_h^* (with respect to the topology of $L_2[0,T;U] \to L_2[0,T;H]$).

Similarly as for T^*, we have:

$$(T_h g,v)_{L_2[0,T;H]} = \int_0^T (T_h g(t),v(t))_H dt$$

Because P_h is an orthogonal projection, the following equality holds:

$$(u_h,v)_H = (u_h,P_h v)_H \qquad \forall v \in H, u_h \in V_h.$$

Hence

$$\int_0^T (T_h g(t),v(t))_H dt = \int_0^T (T_h g(t),P_h v(t))_H dt$$

The semigroup formula for T_h^* is then derived as in section 2.3. Thus we have

$$(3.11) \qquad T_h^* P_h v(t) = B^* \int_t^T S_h^*(\tau-t)P_h v(\tau)d\tau$$

T_h^* can immediately be extended to $L_p[0,T;H]$, $1 \le p \le \infty$ by setting

$$T_h^* v(t) = T_h^* P_h v(t).$$

We now state a preliminary convergence result.

Theorem 3.1. Let $V_h \subseteq D(B^*)$, $0 < h < 1$, be a family of finite-dimensional subspaces of H satisfying properties (3.1) - (3.4) and let A_h, $0 < h < 1$, be a family of finite-dimensional approximations of A satisfying (3.5), (3.6). Let T^* and T_h^* be defined by (2.37) and (3.11) respectively. Then, with $\bar{q}$ as defined in (3.1) - (3.4),

$$|(T^*-T_h^*)v|_{L_p[0,T;U]} \leq ch^{2(1-\bar{q})}\ln\frac{1}{h}\,|v|_{L_p[0,T;H]}, \quad p = 1, \infty.$$

//

Remark. This result is derived in Lasiecka [12]. For the sake of completness we provide its proof in chapter VI.

//

By Theorem 3.1 and Lemma 2.4 we immediately deduce the following theorem.

Theorem 3.2. Let V_h, A_h, $0 < h < 1$, satisfy the same assumptions as in Theorem 3.1. Then

$$|e|_{L_p[0,T;H]} = |u-u_h|_{L_p[0,T;H]} \leq ch^{2(1-\bar{q})}\ln\frac{1}{h}\,|g|_{L_p[0,T;U]}, \quad p = 1,\infty.$$

//

3.4 Application to a Canonical Parabolic Boundary-value Problem

We recall the parabolic problem (2.1):

$$\begin{cases} y_t = A(x,\partial)y & \text{on } Q \\ y(0) = 0 & \text{on } \Omega \\ y|_\Gamma = g & \text{on } \Sigma \end{cases}$$

where $g \in L_p[0,T;L_2(\Gamma)]$, $1 \leq p \leq \infty$ and $A(x,\partial)$ is a uniformly strongly elliptic, second-order differential operator as defined in (1.7).

We shall now see how the abstract approximation scheme of the previous section can be applied to yield the optimal rate of convergence of the Galerkin approximation to problem (2.1).

As we have seen in section 2.3, we can represent (2.1) in integral form, i.e.,

$$y(t) = \int_0^t AS(t-\tau)Dg(\tau)d\tau$$

with the operators A,D,S(t) as defined in that same section 2.3.

We have also seen in section 2.3 that problem (2.1) can be formulated within the context of problem (2.6) by taking $H = L_2(\Omega)$, $U = L_2(\Gamma)$, $B = AD$ (formally) with $q = \frac{3}{4} + \varepsilon$, $\varepsilon > 0$ and $\bar{q} = \frac{3}{4}$.

We shall formulate an approximating scheme for problem (2.1) by introducing a standard class of finite-dimensional subspaces V_h [2] and a family of finite-dimensional operators A_h which comply with the abstract conditions (3.1) - (3.4), (3.4') and (3.5), (3.6).

We recall the definition of an $S_h^{r,k}$-system as given by Babuska and Aziz [2].

Definition. For $0 < h < 1$, let $S_h^{r,k}(\Omega)$ be a linear system of functions defined on Ω, with $r > k \geq 0$, satisfying the following conditions.

(i) $S_h^{r,k}(\Omega) \in H^k(\Omega)$

(ii) for every $v \in H^\ell(\Omega)$, $\ell \geq 0$ and $0 \leq s \leq \min(\ell,k)$, there exists $v_h \in S_h^{r,k}(\Omega)$ such that

$$|v-v_h|_{H^s(\Omega)} \leq ch^\mu |v|_{H^\ell(\Omega)} \tag{3.12}$$

where $\mu = \min(\ell-s, r-s)$ and the constant c is independent of v and h. The function v_h is dependent on v and may be different for different values of s.

//

Definition. An $S_h^{r,k}$-system is said to be regular if the approximating function v_h can be chosen independently of s so that (3.12) holds for all s, with $0 \leq s \leq \min(\ell,k)$.

//

We now recall the Inverse Approximation Property. We say that the system $S_h^{r,k}(\Omega)$ satisfies the inverse approximation property if there exists ε, $0 < \varepsilon \leq k$ such that for every $k - \varepsilon \leq s \leq k$ and every $v_h \in S_h^{r,k}(\Omega)$ we have

$$|v_h|_{H^k(\Omega)} \leq ch^{-(k-s)} |v_h|_{H^s(\Omega)}$$

where c does not depend on h or v_h.

Then we have the following result.

Theorem [2]. Let $S_h^{r,k}(\Omega)$ be an (r,k)-system which satisfies the inverse assumption. Then for every s_1, s_2 with $s_1 \leq s_2 \leq k$ and $v_h \in S_h^{r,k}(\Omega)$, we have

$$|v_h|_{H^{s_2}(\Omega)} \leq ch^{-(s_2-s_1)} |v_h|_{H^{s_1}(\Omega)}$$

where c does not depend on h or v_h (it depends in general on s_1 and s_2). //

We are now in a position to define our family of finite-dimensional subspaces V_h.

Let V_h be a regular $S_h^{2,1}(\Omega)$ system satisfying the inverse assumption. Let P_h be an orthogonal projection from $L_2(\Omega)$ onto V_h. Then

$$(3.13) \qquad |v-P_h v|_s \leq ch^{\alpha-s}|v|_\alpha, \; 0 \leq \alpha \leq 2, \; 0 \leq s \leq 1, \; \alpha-s \geq 0$$

We also require that the subspaces V_h satisfy

$$(3.14) \qquad |\nabla(P_h v-v)| \leq ch^{1/2}|v|_2,$$

$$(3.15) \qquad |\nabla(P_h v-v)| \leq ch^{\alpha}|v|_{(2\alpha+3+\varepsilon)/2}, \qquad 0 \leq \alpha < 1/2,$$

$$(3.16) \qquad \left|\frac{\partial v}{\partial v_A}\right| \leq ch^{-3/2}|v|, \quad \forall v \in V_h.$$

Remark. It is well-known that spaces of splines defined on uniform meshes satisfy the properties of a regular $S_h^{r,k}$-system with the inverse approximation property. For example, linear splines defined on a bounded domain Ω with uniform grid form a regular $S_h^{2,1}(\Omega)$-system with the inverse approximation property. Furthermore, they also satisfy conditions (3.14) - (3.16).

//

As an approximation of A we take an arbitrary operator $A_h: V_h \to V_h$ such that

(3.17) $\quad \|(A^{-1}-A_h^{-1}P_h)x\| \leq ch^2\|x\|,$

(3.18) $\quad \|A_h^{\beta}v_h\| \leq c\|A^{\beta}v_h\|, \qquad 0 \leq \beta \leq 1/2$

and

(3.19) $\quad |(A_hu_h,v_h)| \leq c|u_h|_1|v_h|_1,$

(3.20) $\quad (A_hu_h,u_h) \geq \gamma|u_h|_1^2, \quad \forall u_h,v_h \in V_h.$

At this point we need to make a few remarks.

(i) Most of the well-known approximations A_h to elliptic problems comply with the conditions (3.17) - (3.20). For example, the standard Galerkin approximation A_h, where

(3.21) $\quad (A_hu_h,v_h) = (Au_h,v_h) \quad \forall u_h,v_h \in V_h,$

satisfies (3.17) - (3.20). Other methods, such as Babuska's penalty method [1], and Nitsche's method [17], also satisfy (3.17) - (3.20), and moreover do not require the subspaces V_h to satisfy zero boundary conditions (as the Galerkin method does).

(ii) Using the identifications [7]

$$(3.22) \quad D(A^{\alpha}) = D(A^{*\alpha}) = H_0^{2\alpha}(\Omega), \ 0 \leq \alpha < \frac{3}{4},$$

$$(3.23) \quad D(A) = D(A^*) = H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$(3.24) \quad \|x\|_{2\beta} \leq c\, \|x\|_{D(A^{\beta})} \qquad \text{for } x \in D(A^{\beta}),\ 0 \leq \beta \leq 1,$$

we can see that the conditions (3.13) - (3.16) imply the corresponding abstract conditions (3.1) - (3.4).

In fact, by taking s = 0 in (3.13), we have by (3.24)

$$\|v-P_h v\| \leq ch^{2\alpha}\|v\|_{2\alpha} \leq ch^{2\alpha}\|v\|_{D(A^{\alpha})},\ 0 \leq \alpha \leq 1.$$

Also with $B^* = D^*A^*$ (and thus $q = 3/4+\varepsilon$, $\varepsilon>0$), and recalling from section 2.4 that $D^*A^*v = \frac{\partial v}{\partial \nu_A}$, we have by (3.23) and (3.24):

$$|\nabla(P_h v-v)\cdot\nu_A| = |D^*A^*(P_h v-v)| \le ch^{1/2}|v|_2 \le ch^{1/2}|v|_{D(A^*)},$$

and

$$|\nabla(P_h v-v)\cdot\nu_A| = |D^*A^*(P_h v-v)| \le ch^{2\alpha}|v|_{2\alpha+\frac{3}{2}+2\varepsilon} \le ch^{2\alpha}|v|_{D(A^{*\alpha+\frac{3}{4}+\varepsilon})},$$

$0 \le \alpha \le 1/4-\varepsilon,\ \varepsilon>0$

and finally

$$\left|\frac{\partial v_h}{\partial\nu_A}\right| = |D^*A^*v_h| \le ch^{-3/2}|v_h|.$$

(iii) Condition (3.4') is automatically satisfied since V_h satisfies the inverse assumption.

(iv) It was shown in Lasiecka [11] that with A_h satisfying (3.17) - (3.20), the corresponding semigroup $S_h(t)$ is uniformly analytic, i.e.,

$$|A_h^\beta S_h(t)x_h| \le \frac{c}{t^\beta}|x_h| \qquad \text{uniformly in } h,$$

which is precisely the condition (3.6) with $H = L_2(\Omega)$. This result parallels the well-known result mentioned in Chapter II, which states that if A is a uniformly elliptic operator, then -A is the generator of an analytic semigroup.

(v) Finally, (3.5) follows directly from (3.17).

//

Thus we have proved that the assumptions (3.1) - (3.6) follow from (3.13) - (3.20).

From these remarks we can see that the abstract approximation scheme can be applied to our situation. By taking $H = L_2(\Omega)$, $V = L_2(\Gamma)$, $B = AD$ (formally) with $q = \frac{3}{4} + \varepsilon$, $\varepsilon > 0$, and recalling from (2.39) that

$$B^*v = D^*A^*v = \frac{\partial v}{\partial \nu_A},$$

the algorithm takes the form:

find $y_h(t) \in V_h$ such that

$$(3.25) \quad \begin{cases} (\dot{y}_h(t), v_h)_{L_2(\Omega)} = -(A_h y_h(t), v_h)_{L_2(\Omega)} + \langle g(t), \frac{\partial v_h}{\partial \nu_A}\rangle_{L_2(\Gamma)} \\ y_h(0) = 0 \qquad \text{for all } v_h \in V_h. \end{cases}$$

If A_h is a standard Galerkin approximation (see 3.21), then (3.25) becomes

$$(\dot{y}_h(t), v_h)_{L_2(\Omega)} = -a(y_h(t), v_h) + \langle g(t), \frac{\partial v_h}{\partial \nu_A}\rangle_{L_2(\Gamma)}$$

where $a(u,v)$ is the bilinear form associated to A.

The following convergence result associated with the algorithm (3.25) follows directly from Theorem 3.2.

Theorem 3.3. Let $e(t) = y(t) - y_h(t)$, where $y(t)$ satisfies the parabolic equation (2.1), and $y_h(t)$ is its continuous-time approximation as given in (3.25). Then

$$|e|_{L_p[0,T;L_2(\Omega)]} \leq ch^{1/2} \ln \frac{1}{h} |g|_{L_p[0,T;L_2(\Gamma)]}, \quad p = 1, \infty.$$

//

Remark. In view of the best approximation properties of the subspaces V_h and of the maximal regularity of the solution $y(t)$ ($H_p^{1/2-2\varepsilon, 1/4-\varepsilon}(Q)$ for $p = 1, \infty$ and $q = 3/4+\varepsilon$), it follows that the convergence estimate $O(h^{1/2}\ln\frac{1}{h})$ is optimal.

The same result for $1 < p < \infty$ can be obtained by interpolation. Such a result is, however, not optimal (modulo $\ln\frac{1}{h}$), since the maximal regularity of the solution is $H_p^{1/2, 1/4}(Q)$ for $1 < p < \infty$. The optimal convergence rate $O(h^{1/2})$ has been established and requires more refined techniques found in Lasiecka [12].

//

CHAPTER IV
TIME DISCRETIZATION

4.1 Opening Remarks

In this chapter we shall introduce an approximation scheme which complements the scheme given in chapter III. Whereas the previous scheme approximated the values of the solution for a given value of t and left the time variable t unchanged, in this scheme we shall do the opposite, namely we shall approximate the solution at discrete values of the time variable, and otherwise leave the range of values of the solution unchanged.

In section 4.2 we shall present the basic one-dimensional backward Euler difference scheme, which is used as a basis for an abstract discrete-time approximation scheme.

The next two sections are devoted to the development, justification and statement of the convergence properties of this approximation scheme in the context of an abstract boundary-value problem.

Finally, in the last section 4.5, the abstract convergence results are applied to the canonical example (2.1) and convergence estimates are derived which are seen to be essentially optimal.

4.2 A One-Dimensional Euler Scheme

The time discretization scheme we shall use in this work is based on the backward Euler difference, which we describe here in a simplified context. The general scheme will be given in the next section as an extension of this basic scheme. Consider the one-dimensional initial-value problem

$$(4.1)\quad \begin{cases} \frac{\partial y}{\partial t}(t,x) = -a(x)y(t,x) + f(t,x), & t > 0 \\ y(0,x) = y_o(x) \end{cases}$$

where $x \in J$, J an interval on $\mathbb{R}$, and $a(x) > 0 \ \forall x \in J$.

Let k be the parameter of discretization, with $k > 0$. We let $f_n = f(nk)$, and let y_n be the time-discrete approximation of $y(nk)$. By replacing $\frac{\partial y}{\partial t}(nk)$ with $\frac{1}{k}(y_n - y_{n-1})$, the scheme takes the following form:

find y_n, $n = 1,2, \ldots$, where

$$(4.2)\quad \begin{cases} \frac{1}{k}(y_{n+1} - y_n) = -ay_{n+1} + f_{n+1} & , n = 0,1,\ldots \\ y_o \text{ given.} \end{cases}$$

By solving for y_{n+1}, (4.2) becomes

$$(4.3)\quad \begin{cases} y_{n+1} = \frac{1}{1+ka}(y_n + kf_{n+1}) & , \quad n = 0,1,\ldots \\ y_o \text{ given.} \end{cases}$$

If we set $e_k = 1/(1+ka)$, then the recursive formula (4.3) can be put into the explicit form

$$(4.4) \qquad y_n = e_k^n\, y_o + \sum_{j=1}^{n} e_k^{n+1-j} f_j, \qquad n = 1,2,\ldots$$

to be compared with the formula for the actual solution of (4.1) obtained by the variation of parameter formula:

$$(4.5) \qquad y(t) = e^{-at} y_o + \int_0^t e^{-a(t-\tau)} f(\tau)d\tau, \qquad t > 0$$

For $t = nk$, we have

$$(4.6) \qquad y(nk) = e^{-ank} y_o + \int_0^{nk} e^{-a(nk-\tau)} f(\tau)d\tau$$

$$= (e^{-ak})^n y_o + \sum_{j=1}^{n} (e^{-ak})^{n+1-j} \int_0^k e^{a\tau} f(jk+\tau-k)d\tau$$

The obvious questions about stability and convergence of the scheme will be answered in the more general context of sections 4.2 and 4.3. One can nevertheless notice that $e_k = 1/(1+ka)$ is a second-order approximation in k of e^{-ak} for "small" k. This particular property is the heart of the proof of a result which itself is crucial in establishing stability and convergence (see Lemma 4.1, and its proof in chapter VI).

4.3 Extension of the Scheme to an Abstract Parabolic Boundary-Value Problem

From section 2.2 we recall the abstract problem

$$(4.7)\quad \begin{cases} u_t = Au + Bg & \text{on } D(A^*)' \\ u(0) = u_o \end{cases}$$

where -A is the infinitesimal generator of a strongly continuous analytic semigroup S(t), B is an operator from a Hilbert space U into $D(A^*)'$ with $A^{-q}B \in \mathcal{L}(U \to H)$ for some $0 \leq q \leq 1$, and $g \in L_p[0,T;U]$.

In order to apply the scheme described in the previous section to the problem (4.7), we define the operator

$$P_k\colon\ L_p[0,T;H] \to L_p[0,T;H]$$

by

$$(4.8)\qquad P_k v(t) = \frac{1}{k} \int_{(j-1)k}^{jk} v(\tau)d\tau$$

where T = Nk for some integer N and j is the unique integer such that $(j-1)k < t \leq jk$.

The range of P_k consists of the step functions defined on [0,T] with (discrete) values in H. At this point the following remarks are in order.

(i) $P_k \in \mathcal{L}(L_p[0,T;H] \to L_p[0,T;H])$, $1 \leq p \leq \infty$. Furthermore, $P_k v(t)$ is defined for every $t \in [0,T]$.

(ii) P_k is an orthogonal projection, whch implies that P_k is a selfadjoint operator (with respect to the inner-product in $L_2[0,T;H]$).

(iii) P_k can clearly be defined the same way on $L_p[0,T;U]$ with the same properties.

//

We now define the discrete-time approximation u_k of (4.7) as the solution of the difference equation

$$(4.9)\quad \begin{cases} \frac{1}{k}(u_k((n+1)k)-u_k(nk)) = -Au_k((n+1)k) + BP_k g((n+1)k) \\ u_k(0) \text{ given} \end{cases}$$

which, after solving for $u_k((n+1)k)$, becomes

$$(4.10)\quad \begin{cases} u_k((n+1)k) = (1+kA)^{-1}[u_k(nk) + BP_k g((n+1)k)] \\ u_k(0) \text{ given.} \end{cases}$$

Following the same steps as in section 4.2, we obtain

$$(4.11)\quad u_k(nk) = E_k^n u_o + \sum_{j=1}^{n} E_k^{n+1-j} BP_k g(jk), \qquad n = 1,2,\ldots$$

where $E_k = (1+kA)^{-1} = \frac{1}{k} R(\frac{1}{k}, -A)$.

We can define u_k for other values of t in the obvious way, i.e.,

$$u_k(t) = u_k(nk),\ (n-1)k < t \leq nk,\ n = 1,2,\ldots$$

Again for simplicity and without loss of generality, we assume that $u_o \equiv 0$. We now make the following remarks to justify the validity of formula (4.11).

(i) By (i) and (iii) of the last set of remarks, we deduce that $P_k g(jk)$ is defined for $j = 1,2,\ldots$

(ii) The exponent $(n+1-j)$ of E_k in formula (4.11) is always at least 1; moreover,

$$AE_k \in \mathscr{L}(H \to H) \qquad \text{for } k>0$$

which can be readily seen from the resolvent identity

$$AR(\lambda,-A) = I - \lambda R(\lambda,-A).$$

Therefore, $E_k B \in \mathscr{L}(U \to H)$ since

$$(E_k Bu,v)_H = (AE_k A^{-1} Bu,v)_H \qquad \forall u \in U,\ v \in H.$$

//

Thus the formula (4.11) makes sense in the topology of H.

4.4 Preliminary Results

In order to study the stability and convergence of the abstract scheme, we shall need to establish the approximation properties of P_k. From the orthogonality of P_k, we have

(4.12) $$\|(I-P_k)x\|_{L_p[0,T;H]} \le c\|x\|_{L_p[0,T;H]}, \quad p = 1,\infty.$$

Also,

(4.13) $$\|(I-P_k)x\|_{L_p[0,T;H]} \le ck\|x\|_{H^1_p[0,T;H]}, \quad p = 1,\infty.$$

In fact, for $p = 1, \infty$,

$$\|(I-P_k)x\|_{L_\infty[0,T;H]} = \operatorname*{ess\,sup}_{0\le t\le T} \|x(t)-P_kx(t)\|_H$$

$$= \max_{j=1,\dots,N} \{\operatorname*{ess\,sup}_{(j-1)k<t\le jk} \|\frac{1}{k}\int_{(j-1)k}^{jk} (x(t)-x(\tau))d\tau\|_H\}.$$

By the Mean Value Theorem,

$$\|(I-P_k)x\|_{L_\infty[0,T;H]} \le \max_{j=1,\dots,N} \{\frac{1}{k}\int_{(j-1)k}^{jk} k \sup_{\xi}\|\dot{x}(\xi)\|_H d\tau\}$$

$$\le k\,\|\dot{x}\|_{L_\infty[0,T;H]}$$

$$\le k\,\|x\|_{H^1_\infty[0,T;H]}.$$

For $p = 1$, we have similarly

$$\|(I-P_k)x\|_{L_1[0,T;H]} = \int_0^T\|x(t)-P_kx(t)\|_H dt$$

$$= \sum_{j=1}^{N}\int_{(j-1)k}^{jk} \|x(t)-P_kx(t)\|_H dt$$

$$= \sum_{j=1}^{N} \int_{(j-1)k}^{jk} \left| \frac{1}{k} \int_{(j-1)k}^{jk} (x(t)-x(\tau))d\tau \right|_H dt$$

$$\leq \frac{1}{k} \sum_{j=1}^{N} \int_{(j-1)k}^{jk} \int_{(j-1)k}^{jk} \left| \int_{\tau}^{t} \dot{x}(\xi)d\xi \right|_H d\tau dt$$

$$\leq \frac{1}{k} \sum_{j=1}^{N} \int_{(j-1)k}^{jk} \int_{(j-1)k}^{jk} \left| \int_{\tau}^{t} |\dot{x}(\xi)|_H d\xi \right| d\tau dt$$

$$\leq \frac{1}{k} \sum_{j=1}^{N} \int_{(j-1)k}^{jk} \int_{(j-1)k}^{jk} \int_{(j-1)k}^{jk} |\dot{x}(\xi)|_H d\xi d\tau dt$$

$$\leq k \, |\dot{x}|_{L_1[0,T;H]}$$

$$\leq k \, |x|_{H_1^1[0,T;H]} .$$

A well-known interpolation theorem [13, Vol. I, p. 27] then enables us to conclude that

$$(4.14) \qquad \|(I-P_k)x\|_{L_p[0,T;H]} \leq ck^{\alpha}|x|_{H_p^{\alpha}[0,T;H]} , \quad p=1, \infty, \ 0 \leq \alpha \leq 1$$

for $x \in H_p^1[0,T;H]$.

The estimate (4.14) can then be extended by density to $x \in H_p^{\alpha}[0,T;H]$.

We shall make strong use of (4.14) in the proof of Theorem 4.2 and also in that of Theorem 5.2.

We now state another result which we will need in order to establish stability and convergence of the abstract approximation scheme.

Lemma 4.1. Let A: H→H be an arbitrary closed linear operator such that A is the infinitesimal generator of a strongly continuous analytic semigroup S(t). Without loss of generality, we assume that

$$\sigma(A) \subseteq \Sigma = \{\lambda : \phi < |\arg\lambda| \leq \pi \text{ for some } \phi > \tfrac{\pi}{2}\} \setminus \{0\}$$

Let $k > 0$ be fixed and let $E_k = \frac{1}{k} R(\frac{1}{k}, A)$, where $R(\lambda, A)$ is the resolvent of A. Then

$$(4.15) \qquad |(E_k^n - S(nk))x|_H \leq \frac{c}{n} |x|_H, \qquad n = 1,2,\ldots$$

and

$$(4.16) \qquad |A(E_k^n - S(nk))x|_H \leq \frac{c}{nk} |x|_H, \qquad n = 1,2,\ldots$$

The constant c does not depend on n or k. It depends, however, on ϕ, as can be seen in the proof. The proof of Lemma 4.1 is given in Chapter VI.

//

In order to state our theorem on stability and convergence of the discrete-time approximation, we need to define an input-solution operator that we can compare with T, i.e.,

$$T_k\colon\ L_p[0,T;U] \to L_p[0,T;H],$$

where

$$(4.17)\qquad T_k g(t) = u_k(nk) = k \sum_{j=1}^{n} E_k^{n+1-j} B(P_k g)(jk) \text{ for } (n-1)k<t\leq nk.$$

Since the theorem will be stated and proved in terms of adjoint operators, we derive the formula for T_k^*. First, we note that

$$(T_k g,v)_{L_2[0,T;H]} = (T_k g,P_k v)_{L_2[0,T;H]}$$

by orthogonality of P_k. Then,

$$(T_k g,P_k v)_{L_2[0,T;H]} = \int_0^T (T_k g(t),P_k v(t))_H dt$$

$$= \sum_{n=1}^{N} \int_{(n-1)k}^{nk} (T_k g(t),P_k v(t))_H dt\,.$$

By definition of T_k and P_k, we have

$$T_k g(t) = T_k g(nk) \text{ and } P_k v(t) = P_k v(nk) \text{ for } (n-1)\ k < t \leq nk.$$

Therefore,

$$(T_k g,P_k v)_{L_2[0,T;H]} = \sum_{n=1}^{N} \int_{(n-1)k}^{nk} (T_k g(nk),P_k v(nk))_H dt$$

$$= k \sum_{n=1}^{N} (T_k g(nk),P_k v(nk))_H$$

$$= k \sum_{n=1}^{N} (k \sum_{j=1}^{n} E_k^{n+1-j} B(P_k g)(jk), P_k v(nk))_H$$

$$= k \sum_{n=1}^{N} k \sum_{j=1}^{n} (E_k^{n+1-j} B(P_k g)(jk), P_k v(nk))_H$$

$$= k^2 \sum_{n=1}^{N} \sum_{j=1}^{n} (P_k g(jk), B^* E_k^{*\,n+1-j} P_k v(nk))_U$$

By reversing the order of summation,

$$(T_k g, P_k v)_{L_2[0,T;H]} = k^2 \sum_{j=1}^{N} \sum_{n=j}^{N} (P_k g(jk), B^* E_k^{*\,n+1-j} P_k v(nk))_U$$

$$= k \sum_{j=1}^{N} (P_k g(jk), kB^* \sum_{n=j}^{N} E_k^{*\,n+1-j} P_k v(nk))_U$$

$$= k \sum_{j=1}^{N} (\frac{1}{k} \int_{(j-1)k}^{jk} g(t)dt, T_k^* P_k v(jk))_U$$

$$= \sum_{j=1}^{N} \int_{(j-1)k}^{jk} (g(t), T_k^* P_k v(t))_U dt$$

$$= (g, T_k^* P_k v)_{L_2[0,T;U]}$$

Therefore

$$(4.18)\qquad T_k^* P_k v(t) = kB^* \sum_{n=j}^{N} E_k^{*\,n+1-j} P_k v(nk) \quad \text{for } (j-1)k < t \le jk.$$

T_k^* extends to the whole of $L_p[0,T;H]$ in the obvious manner.

We have the following preliminary theorem.

Theorem 4.2. With T^* and T_k^* as defined in (2.37) and (4.18) respectively, we have

$$(4.19)\qquad |(T^*-T_k^*)v|_{L_p[0,T;U]} \le ck^{1-q-\varepsilon-\alpha}|v|_{L_p[0,T;D(A^{*-\alpha})]}$$

for $1 \le p \le \infty$, $0 \le \alpha \le 1-q$, $\varepsilon > 0$, where $A^{-q}B \in \mathscr{L}(U\to H)$ for some q, $0 \le q \le 1$.

//

The proof of Theorem 4.2 will be given in chapter VI.

We are now in a position to state our stability and convergence theorem.

Theorem 4.3. Let $e(t) = u(t) - u_k(t)$, where u and u_k are solutions to the abstract problems (2.6) and (4.9) respectively, and given by the respective formulas (2.8) and (4.11) with $u_0 \equiv 0$. Then

$$|e|_{L_p[0,T;D(A^\alpha)]} \le ck^{1-q-\varepsilon-\alpha}|g|_{L_p[0,T;U]}$$

for $1 \leq p \leq \infty$, $0 \leq \alpha \leq 1-q$, $\varepsilon>0$, where $A^{-q}B \in \mathcal{L}(U \to H)$ for some q, $0 \leq q \leq 1$.

//

Proof of Theorem 4.3. We note first of all that $u(t) = Tg(t)$ and $u_k(t) = T_k g(t)$, thus $e(t) = (T-T_k)g(t)$. Theorem 4.3 then follows immediately by application of Lemma 2.4.

//

4.5 Application to a Canonical Example

Similarly as in the previous chapters, we apply our abstract result to the case of the canonical problem (2.1), i.e.

$$\begin{cases} y_t = -A(x,\partial)y & \text{on } Q \\ y(0,x) = 0 & \text{on } \Omega \\ y|_\Gamma = g & \text{on } \Sigma \end{cases}$$

As we have seen earlier in chapter II, (2.1) can be expressed as an abstract boundary-value problem with $H = L_2(\Omega)$, $U = L_2(\Gamma)$, $B = AD$ (formally) with $A^{-q}B \in \mathcal{L}(U \to H)$ and $q = 3/4 + \varepsilon$, $\varepsilon > 0$.

Then as a discrete-time approximation of (2.22), i.e.,

$$Lg(t) = A \int_0^t S(t-\tau)Dg(\tau)d\tau,$$

we define (see formula (4.17))

$$(4.21) \qquad L_k g(t) = k \sum_{j=1}^{n} AE_k^{n+1-j} P_k(Dg)(jk), \quad (n-1)k < t \le nk.$$

By taking $\alpha = 0$ in Theorem 4.3, we have the following convergence result for problem (2.1).

Theorem 4.4. With L and L_k given by (2.22) and (4.21) respectively, we have

$$\|(L-L_k)g\|_{L_p[0,T;L_2(\Omega)]} \le ck^{1/4-\varepsilon}\|g\|_{L_p[0,T;L_2(\Gamma)]}, \quad 1 \le p \le \infty, \ \varepsilon > 0$$

//

Remark. By comparing the rate of convergence in Theorem 4.4 ($O(k^{1/4-\varepsilon})$) with the time regularity of the solution to the problem (2.1) (which is $H_p^{1/4-\varepsilon}[0,T;L_2(\Omega)]$ for $p = 1, \infty$, and $H_p^{1/4}[0,T;L_2(\Omega)]$ for $1 < p < \infty$), one can see that such a rate of convergence is essentially optimal for $p = 1, \infty$, and almost optimal (modulo ε) for $1 < p < \infty$. We shall see in the next chapter that by combining the two discretization schemes (continuous-time and discrete-time), the rates of convergence obtained for the fully discrete approximation are $O(h^{1/2}\ln\frac{1}{h})$ and $O(k^{1/4}\ln\frac{1}{k})$ for $1 \le p \le \infty$, which are optimal for $p = 1, \infty$ and almost optimal (modulo $\ln\frac{1}{h}$ and $\ln\frac{1}{k}$) for $1 < p < \infty$.

Finally, the stability of L_k is clear by taking $\alpha = 1-q = 1/4-\varepsilon$. Then

$$\|L_k g\|_{L_p[0,T;D(A^{1/4-\varepsilon})]} \le \|Lg\|_{L_p[0,T;D(A^{1/4-\varepsilon})]} + \|(L-L_k)g\|_{L_p[0,T;D(A^{1/4-\varepsilon})]} \le c\|g\|_{L_p[0,T;D(A^{1/4-\varepsilon})]} \quad \text{for } 1 \le p \le \infty.$$

TABLE 1. Values of $\log_{10}(10^2 e_{hk})$ for solution u_1 in the norm of $L_1[0,T;L_2\ \Omega]$.

<--- $\log_2 \frac{1}{k}$ --->

$\log_2 \frac{1}{h}$ =	1	2	3	4	5	6	7	8	9	10
1	1.58	1.59	1.59	1.59	1.59	1.59	1.59	1.59	1.59	1.59
2	1.45	1.45	1.45	1.45	1.45	1.46	1.46	1.46	1.46	1.46
3	1.32	1.32	1.30	1.30	1.30	1.30	1.30	1.30	1.30	1.30
4	1.18	1.18	1.18	1.15	1.15	1.15	1.15	1.15	1.15	1.15
5	1.04	1.04	1.04	1.00	1.00	1.00	1.00	1.00	1.00	1.00
6	0.95	0.91	0.88	0.87	0.86	0.86	0.86	0.86	0.86	0.86
7	0.86	0.80	0.75	0.72	0.72	0.71	0.71	0.71	0.71	0.71
8	0.79	0.70	0.63	0.59	0.56	0.56	0.56	0.56	0.56	0.56
9	0.73	0.62	0.52	0.46	0.43	0.41	0.41	0.41	0.41	0.41
10	0.71	0.57	0.43	0.34	0.28	0.28	0.26	0.26	0.26	0.26

TABLE 2. Values of $\log_{10}(10^2 e_{hk})$ for solution u_1 in the norm of $L_\infty[0,T;L_2(\Omega)]$.

	<--- $\log_2 \frac{1}{k}$ --->									
$\log_2 \frac{1}{h}$ =	1	2	3	4	5	6	7	8	9	10
1	1.59	1.59	1.59	1.59	1.59	1.59	1.59	1.59	1.59	1.59
2	1.45	1.45	1.46	1.46	1.46	1.46	1.46	1.46	1.46	1.46
3	1.30	1.30	1.30	1.30	1.30	1.30	1.30	1.30	1.30	1.30
4	1.18	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15
5	1.04	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
6	0.90	0.87	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86
7	0.78	0.72	0.72	0.71	0.71	0.71	0.71	0.71	0.71	0.71
8	0.68	0.59	0.57	0.56	0.56	0.56	0.56	0.56	0.56	0.56
9	0.60	0.48	0.43	0.41	0.41	0.41	0.41	0.41	0.41	0.41
10	0.56	0.38	0.30	0.26	0.26	0.26	0.26	0.26	0.26	0.26

TABLE 3. Values of $\log_{10}(10^2 e_{hk})$ for solution u_2 in the norm of $L_1[0,T;L_2\ \Omega]$.

<--- $\log_2 \frac{1}{k}$ --->

$\log_2 \frac{1}{h}$ =	1	2	3	4	5	6	7	8	9	10
1	1.62	2.53	1.77	1.68	1.65	1.64	1.63	1.63	1.63	1.63
2	2.08	2.49	1.64	1.53	1.49	1.49	1.48	1.48	1.48	1.48
3	2.23	2.53	1.57	1.40	1.34	1.32	1.32	1.32	1.32	1.32
4	2.26	2.56	1.53	1.28	1.20	1.18	1.18	1.18	1.18	1.15
5	2.26	2.56	1.51	1.20	1.08	1.04	1.00	1.00	1.00	1.00
6	2.26	2.56	1.49	1.15	0.97	0.90	0.87	0.86	0.86	0.86
7	2.26	2.56	1.48	1.11	0.89	0.78	0.73	0.72	0.71	0.71
8	2.26	2.56	1.48	1.08	0.83	0.67	0.60	0.57	0.56	0.56
9	2.26	2.56	1.48	1.08	0.78	0.58	0.48	0.43	0.41	0.41
10	2.26	2.56	1.46	1.08	0.75	0.52	0.36	0.30	0.28	0.26

TABLE 4. Values of $\log_{10}(10^2 e_{hk})$ for solution u_2 in the norm of $L_\infty[0,T;L_2\,\Omega]$.

<--- $\log_2 \frac{1}{k}$ --->

$\log_2 \frac{1}{h} =$	1	2	3	4	5	6	7	8	9	10
1	1.60	2.48	1.71	1.63	1.61	1.61	1.61	1.61	1.61	1.61
2	2.04	2.41	1.57	1.48	1.46	1.46	1.46	1.46	1.46	1.46
3	2.20	2.46	1.46	1.32	1.32	1.32	1.30	1.30	1.30	1.30
4	2.23	2.48	1.41	1.20	1.18	1.15	1.15	1.15	1.15	1.15
5	2.23	2.48	1.38	1.08	1.00	1.00	1.00	1.00	1.00	1.00
6	2.23	2.48	1.34	0.97	0.87	0.86	0.86	0.86	0.86	0.86
7	2.23	2.48	1.34	0.89	0.73	0.71	0.71	0.71	0.71	0.71
8	2.23	2.48	1.34	0.83	0.60	0.56	0.56	0.56	0.56	0.56
9	2.23	2.48	1.32	0.80	0.49	0.41	0.41	0.41	0.41	0.41
10	2.23	2.48	1.32	0.79	0.43	0.28	0.26	0.26	0.26	0.26

TABLE 5.

(a) Values of the solution at $T = \frac{1}{2}$ calculated for the boundary input $g_0(t) = 1$, $g_1(t) = 1$, $0 < t \leq \frac{1}{2}$.

x =	1/8	2/8	3/8	4/8	5/8	6/8	7/8
N = 8	.98121	.96529	.95467	.95094	.95467	.96529	.98121
N = 16	.99350	.98798	.98430	.98301	.98430	.98798	.99350
N = 32	.99582	.99227	.98991	.98908	.98991	.99227	.99582
N = 64	.99633	.99322	.99114	.99041	.99114	.99322	.99633
N = 128	.99646	.99345	.99144	.99074	.99144	.99345	.99646
N = 256	.99649	.99351	.99152	.99082	.99152	.99351	.99649
N = 512	.99649	.99352	.99153	.99084	.99153	.99352	.99649
N = 1024	.99650	.99352	.99154	.99084	.99154	.99352	.99650

(b) Values of the solution at $T = \frac{1}{2}$ calculated for the boundary input $g_0(t) = 1$, $0 < t \leq \frac{1}{2}$ and

$g_1(t) = 0$, $0 < t \leq 1/8$, $\frac{1}{4} < t \leq 3/8$

$g_1(t) = 1$, $1/8 < t \leq \frac{1}{4}$, $3/8 < t \leq \frac{1}{2}$

x =	1/8	2/8	3/8	4/8	5/8	6/8	7/8
N = 8	.93124	.86743	.81382	.77641	.76303	.78517	.86121
N = 16	.91927	.83682	.75243	.66989	.60198	.58020	.67304
N = 32	.92285	.83949	.74438	.63421	.51243	.40516	.42378
N = 64	.92533	.84376	.74900	.63615	.50275	.35225	.23835
N = 128	.92598	.84490	.75039	.63751	.50369	.34958	.18274
N = 256	.92614	.84519	.75074	.63785	.50398	.34977	.17939
N = 512	.92618	.84526	.75083	.63794	.50405	.34982	.17940
N = 1024	.92619	.84528	.75085	.63796	.50407	.34983	.17941

CHAPTER V
FULL TIME AND SPACE DISCRETIZATION

5.1 Opening Remarks

In this chapter we shall introduce a full discretization scheme in the context of an abstract boundary-value problem.

In section 5.2, the procedure of full discretization will be described. The relevance of the order in which the schemes are applied will also be discussed.

In section 5.3, we shall state our main convergence result in the context of the abstract problem (2.6).

Finally, in section 5.4, we shall apply these abstract convergence estimates to the canonical parabolic equation (2.1) and we shall see that optimal rates of convergence are obtained.

5.2 The Abstract Discretization Scheme

In this section we shall see how the scheme is constructed as a combination of the two semidiscrete schemes, i.e. the continuous-time and the discrete-time approximation schemes as described in chapters III and IV respectively.

A natural way to formulate the fully discrete scheme is to use the differential form of the schemes, more precisely the starting point is the weak form of the abstract problem (2.6) as stated in (2.7), which we recall here:

$$\begin{cases} (\dot{u}(t),v)_H = -(Au(t),v)_H + \langle g(t),B^*v\rangle_U \quad \text{for all } v \in D(B^*) \\ u(0) = u_o \end{cases}$$

One way to proceed is to consider the continuous-time discretization scheme applied to (2.7), which we recall here as stated in (3.9):

find $u_h(t) \in V_h$ such that

$$\begin{cases} (\dot{u}_h(t),v_h)_H = -(A_h u_h(t),v_h)_H + \langle g(t),B^*v_h\rangle_U \quad \text{for all } v_h \in V_h, \\ u_h(0) \text{ given} \end{cases}$$

where A_h: $V_h \to V_h$ approximates the operator A.

We also recall the discrete-time approximation scheme as stated in (4.10):

$$\begin{cases} \frac{1}{k}(u_{n+1}-u_n) = -Au_{n+1} + Bg_{n+1}, \qquad n = 0,1,\ldots \\ u_o \text{ given} \end{cases}$$

where $g_n = (P_k g)(nk)$, and u_n is the approximate of $u(nk)$.

By applying the later scheme to (3.9), we obtain

$$\begin{cases} \frac{1}{k}\,(u_{hk}^{(n+1)} - u_{hk}^{(n)}, v_h)_H = -(A_h u_{hk}^{(n+1)}, v_h)_H + \langle g_{n+1}, B^* v_h \rangle_U \\ u_{hk}^{(o)} \text{ given} \qquad\qquad \text{for all } v_h \in V_h \end{cases}$$

where $u_{hk}^{(n)}$ approximates $u_h(nk)$ (and consequently $u(nk)$). By factoring $u_{hk}^{(n+1)}$, one then has

$$(5.1) \quad \begin{cases} ((1+kA_h)u_{hk}^{(n+1)}, v_h)_H = (u_{hk}^{(n)}, v_h)_H + k \langle g_{n+1}, B^* v_h \rangle_U \\ u_{hk}^{(o)} \text{ given} \qquad\qquad \text{for all } v_h \in V_h \end{cases}$$

If the continuous-time discretization scheme is viewed as a projection scheme which restricts the values of the variable to a finite-dimensional space V_h of the space of values H, then the process of full discretization can be carried out in reverse order.

In fact, by applying the scheme (4.10) to the abstract problem (2.6), we get

$$\begin{cases} \frac{1}{k}(u_{n+1} - u_n) = -Au_{n+1} + Bg_{n+1} \quad \text{in } D(A^*)', \ n = 0,1,\ldots \\ u_o \text{ given} \end{cases}$$

where $g_n = (P_k g)(nk)$ and u_n is the approximation of $u(nk)$. Solving for u_{n+1}, we obtain

$$\begin{cases} (1 + kA)u_{n+1} = u_n + kBg_{n+1} \qquad \text{in } D(A^*)', \ n = 0,1,\ldots \\ u_o \text{ given} \end{cases}$$

which in weak form becomes

$$\begin{cases} ((1+kA)u_{n+1},v)_H = (u_n,v)_H + k\, \langle g_{n+1},B^*v\rangle_U \quad \text{for all } v \in D(B^*) \\ u_o \text{ given} \end{cases}$$

Then, as we have indicated above, the continuous-time discretization scheme can be carried out (in this case the term "continuous-time" is very inappropriate) by simply restricting the values of u_n and v to a finite-dimensional space $V_h \subseteq D(B^*)$ and by replacing A with its approximation $A_h: V_h \to V_h$. We then obtain exactly the scheme (5.1).

This scheme is carried out in practice by on the one hand selecting a certain number M of basis functions for the space V_h. This number determines the value of the parameter of discretization h (usually $\frac{1}{M}$). A popular choice is to select basis functions with local supports in such a way that the inner-product matrices (the so-called mass and stiffness matrices) have a band structure or at least a sparse structure. These functions form what is called a finite element basis.

On the other hand, one has a finite-difference scheme such as the weak formulation of (4.10), i.e.,

$$\begin{cases} ((1+kA)u_k((n+1)k),v)_H = (u_k(nk),v)_H + k\, \langle P_k g((n+1)k),B^*v\rangle_U \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in D(B^*), \\ u_k(0) \text{ given.} \end{cases}$$

As we have just seen, the order in which the semidiscrete schemes are applied is immaterial and actually has little meaning. It is more appropriate in this situation to view the fully discrete scheme as a juxtaposition or simultaneous application rather than a successive application of the two semidiscrete schemes. After assembling the different matrices and vectors, one is faced with a sequence of linear systems of algebraic equations to be solved, which in theory is an elementary task (but not always in practice!). The finite element method is well-known for its advantages over standard finite difference algorithms in most cases, such as increased accuracy and flexibility of application. We shall use the finite element method in chapter VII, where it will be carried out in the canonical case of the heat equation.

Remark. In the case of a Galerkin approximation, the single step of restricting u_n and v to the subspace V_h is sufficient since by definition of the Galerkin approximation A_h,

//

$$(A_h u_h, v_h)_H = (Au_h, v_h)_H \qquad \forall u_h, v_h \in V_h.$$

5.3 Estimation of the Error

Whereas the formulation and numerical implementation of the fully discrete approximation are most naturally carried out by using the differential forms of the problems, it will be to our advantage, as we shall see, to estimate the error of discretization by examining and comparing the semigroup formulas corresponding to the different discretization schemes.

Our first goal in this section is to derive a (discrete) semigroup formula for the fully discrete approximation to the abstract problem

(2.6). As in section (5.2), such a formula can be derived in two ways, which, as we have pointed out in the previous section, are merely two ways of describing the same idea.

As for the first way, we recall from section 3.2 that the semigroup formula for the continuous-time approximation of problem (2.6), given by (3.10), is

$$u_h(t) = \int_0^t S_h(t-\tau)P_h(Bg(\tau))dt.$$

The discrete-time algorithm applied to the solution of problem (2.6) can be also applied to $u_h(t)$ without modification, and thus we obtain a formula similar to (4.11), i.e.:

$$(5.2) \qquad u_{hk}(nk) = E_{hk}^n P_h u_o + k \sum_{j=1}^{n} E_{hk}^{n+1-j} P_h B(P_k g)(jk), \quad n = 1,2,\ldots$$

where $E_{hk} = \frac{1}{k} R(\frac{1}{k}, -A_h)$ and A_h is an approximation of A. As in (4.11), $u_{hk}(t)$ is defined in the obvious way for the other values of t.

The second way to derive a formula for the fully discrete approximation u_{hk} is to start with the formula for the discrete-time approximation u_k of the solution to problem (2.6), i.e.

$$u_k(nk) = E_k^n u_o + k \sum_{j=1}^{n} E_k^{n+1-j} B(P_k g)(jk), \qquad n = 1,2,\ldots$$

The fully discrete approximation u_{hk} is then obtained by applying the orthogonal projection P_h to the input $B(P_k g)(jk)$ and to the initial data

u_o and by replacing A with its approximation A_h. It can be readily seen that this procedure yields precisely (5.2).

We are concerned with the estimation of $u-u_{hk}$ in the norms $L_p[0,T;H]$ for input $g \in L_p[0,T;U]$, $p = 1, \infty$. To this effect we define the operator $T_{hk}: L_p[0,T;U] \to L_p[0,T;H]$ by

$$(5.3) \quad T_{hk}g(t) = u_{hk}(t) = k \sum_{j=1}^{n} E_{hk}^{n+1-j} P_h B(P_k g)(jk), \qquad n = 1,2,\ldots,$$

where we assume without loss of generality that $u_o = 0$.

It is clear that T_{hk} is well-defined in the topology of $L_p[0,T;H]$ for g in $L_p[0,T;U]$, and this for the same reasons given for T_k.

Since we shall estimate the error of discretization with the adjoints, we need to derive the formula for T_{hk}^*. We have

$$(T_{hk}g,v)_{L_2[0,T;H]} = \int_0^T (T_{hk}g(t),v(t))_H dt$$

$$= \int_0^T (T_{hk}g(t),P_h v(t))_H dt$$

since $T_{hk}g(t) \in V_h$ for all $t > 0$. From this point on, the derivation of T_{hk}^* follows exactly that of T_k^*. Thus the formula for T_{hk}^* is

$$(5.4) \quad T_{hk}^* P_h v(jk) = kB^* \sum_{n=j}^{N} E_{hk}^{*n+1-j} P_k((P_h v)(nk)), \; j = 1,2,\ldots$$

Remark. The order in which P_h and P_k appear in the formulas (5.3) and (5.4) is irrelevant since P_h and P_k commute. Therefore we can extend T_{hk}^* to all of $L_p[0,T;H]$ in the obvious way.

//

We are now in a position to state a basic convergence result.

Theorem 5.1. For $0 < h < 1$, let $V_h \subseteq D(B^*)$ be a family of finite-dimensional subspaces of H satisfying properties (3.1) - (3.4), and let A_h be a family of approximations of the operator A satisfying (3.5) and (3.6). Let T^* and T_{hk}^* be defined by (2.37) and (5.4) respectively. Then

$$\|(T^*-T_{hk}^*)v\|_{L_p[0,T;U]} \le c(h^{2(1-\bar{q})}\ln\frac{1}{h} + h^{-2\bar{q}}k \ln\frac{1}{k})\|v\|_{L_p[0,T;H]}$$

for $p = 1, \infty$, where $\bar{q} \le q$ and $A^{-q}B \in \mathcal{L}(U \to H)$ for some $0 \le q \le 1$. The parameters of discretization h and k are subject to the restriction $k \le ch^2$ with $c > 0$ arbitrarily chosen.

//

The restriction $k \le ch^2$ can be removed by imposing stronger approximation properties on the subspaces V_h. In fact, we have

Theorem 5.2. For $0 < h < 1$, let $V_h \subseteq D(A^*)$ be a family of finite-dimensional subspaces of H satisfying the properties (3.1) - (3.3) and (3.4'), and let A_h be as in Theorem 5.1 a family of approximations of the operator A satisying (3.5), (3.6). Let T^* and T_{hk}^* be defined by (2.37) and (5.4) respectively.

Then

$$\|(T^*-T^*_{hk})v\|_{L_p[0,T;U]} \leq c(h^{2(1-q)} + k^{1-q})\|v\|_{L_p[0,T;H]}$$

for $p = 1, \infty$, where $A^{-q}B \in \mathcal{L}(U \to H)$ for some q, $0 \leq q \leq 1$. There is no restriction on k or h.

//

The estimates stated in Theorems 5.1 and 5.2 are also valid by duality for $T-T_{hk}$. In fact, from Theorems 5.1 and 5.2, it follows immediately by Lemma 2.4 that

$$(5.5)\qquad \|(T-T_{hk})g\|_{L_p[0,T;H]} \leq c(h^{2(1-\bar{q})}\ln\frac{1}{h} + k^{(1-\bar{q})}\ln\frac{1}{k})\|g\|_{L_p[0,T;U]}$$

for $p = 1, \infty$, by making use of the restriction $k \leq ch^2$ and with the assumptions of Theorem 5.1. Also, we have

$$(5.6)\qquad \|(T-T_{hk})g\|_{L_p[0,T;H]} \leq c(h^{2(1-q)} + k^{1-q})\|g\|_{L_p[0,T;U]}$$

for $p = 1, \infty$, with the assumptions of Theorem 5.2, and without restriction on k or h.

We now make the following remarks.

(i) The estimates stated in Theorem 5.1 and 5.2 can be extended to the case $1 < p < \infty$ by interpolation.

(ii) In Theorem 5.1, the subspaces V_h are taken in $D(B^*)$, whereas in Theorem 5.2, they are required to lie inside $D(A^*)$. This latter condition is clearly stronger and is necessary in order to prove the auxiliary result $A^*A_h^{*-1}P_h \in \mathcal{L}(H \to H)$ (see section 6.6) on which the estimation of $T^*-T^*_{hk}$ is based. In the next section we shall see what this stronger condition means in the context of the canonical parabolic problem (2.1).

5.4 Application to a Canonical Example

The abstract scheme (5.1) and its corresponding solution (5.3), as well as the results (5.5) and (5.6), can be specialized directly to the case of our parabolic problem (2.1) by taking $H = L_2(\Omega)$, $U = L_2(\Gamma)$, $B = AD$ (formally) with $A^{-q}B \in \mathcal{L}(U \to H)$ for $q = \frac{3}{4} + \varepsilon$, $\varepsilon>0$, and $\bar{q} = \frac{3}{4}$.

Also the abstract scheme (5.1), specialized to this particular context, takes the form

$$(5.7)\quad \begin{cases} ((1 + kA_h)y_{hk}^{(n+1)}, v_h)_{L_2(\Omega)} \\ \quad = (y_{hk}^{(n)}, v_h)_{L_2(\Omega)} + k < g_{n+1}, \frac{\partial v_h}{\partial \nu_A} >_{L_2(\Gamma)} \qquad n = 0,1,\ldots \\ \text{for all } v_h \in V_h \\ y_{hk}^{(o)} = 0 \end{cases}$$

In fact, as our main result we have the following theorem

Theorem 5.3. Let $V_h \subseteq D(D^*A^*)$ be a regular $S_h^{2,1}(\Omega)$ system satisfying the inverse assumption, as well as properties (3.14) - (3.16). Also let A_h be an arbitrary approximation of A satisfying (3.17) - (3.20). If y(t) is the solution of the parabolic problem (2.1) and $y_{hk}(t)$ is the solution of the fully discrete scheme (5.7), then $e(t) = y(t) - y_{hk}(t)$ satisfies the inequality

$$|e|_{L_p[0,T;L_2(\Omega)]} \leq c(h^{1/2}\ln\frac{1}{h} + k^{1/4}\ln\frac{1}{k})|g|_{L_p[0,T;L_2(\Gamma)]}$$

for $1 \leq p \leq \infty$ with the restriction $k \leq ch^2$ for $c > o$ arbitrarily chosen.

Similarly, we have corresponding to Theorem 5.2 the following result.

Theorem 5.4. Let V_h, A_h satisfy the same assumptions as in Theorem 5.3, with the difference that we now require that $V_h \subseteq D(A^*)$. Also, let y(t) be the solution of the parabolic problem (2.1) and $y_{hk}(t)$ the solution of the fully discrete scheme (5.7). Then $e(t) = y(t) - y_{hk}(t)$ satisfies the inequality

$$|e|_{L_p[0,T;L_2(\Omega)]} \leq c(h^{1/2-2\varepsilon} + k^{1/4-\varepsilon})|g|_{L_p[0,T;L_2(\Gamma)]}$$

for $1 \leq p \leq \infty$, and without any restriction on k or h.

//

We now make the following comments.

(i) In view of the best approximation properties of V_h and the maximal regularity of the solution y to the parabolic problem (2.1), the estimates $O(h^{1/2} \ln \frac{1}{h})$ and $O(k^{1/4} \ln \frac{1}{k})$ are almost optimal (modulo $\ln \frac{1}{h}$ and $\ln \frac{1}{k}$) for $1 < p < \infty$, and are optimal for $p = 1, \infty$.

(ii) If we recall the rates of convergence obtained for each semidiscrete approximation scheme ($O(h^{1/2} \ln \frac{1}{h})$ and $O(k^{(1/4-\varepsilon)})$, we see that in Theorem 5.3 essentially a power ε of k has been regained in the estimates.

The slight difference in these rates of convergence reflects the methods used in proving each estimate. In the case of the fully discrete approximation, use is made of the inverse approximation property (3.4) of the subspaces V_h to "kill" the ε in $O(k^{(1/4-\varepsilon)})$. This improvement is obtained at the price of restricting the values of the parameters of discretization h and k by the condition $k \leq ch^2$.

As we have seen, this restriction can be removed by imposing stronger approximation properties on the subspaces V_h. Unfortunately, the optimal rates of convergence $O(h^{1/2} \ln \frac{1}{h})$ and $O(k^{1/4} \ln \frac{1}{k})$ are lost in the process and we are left with $O(h^{(1/2-2\varepsilon)})$ and $O(k^{(1/4-\varepsilon)})$. In the case $p = 1, \infty$, these latter rates of convergence can nevertheless be viewed as essentially optimal since the regularity of the solution of (2.1) is $H_p^{1/2-2\varepsilon, 1/4-\varepsilon}(Q)$.

From the point of view of numerical computations, the restriction $k \leq ch^2$ is far more significant than the loss of "ε" in the rates of convergence. In the case of Theorem 5.4 another significant restriction is $V_h \subseteq D(A^*)$. It is known that linear splines defined on uniform meshes are in $D(D^*A^*)$ but not in $D(A^{*q})$ for $q > \frac{3}{4}$. Therefore, whenever linear splines are used Theorem 5.4 does not apply and the restriction $k \leq ch^2$ is a necessity.

As for the case $1 < p < \infty$, the rates of convergence $O(h^{1/2})$ and $O(k^{(1/4-\varepsilon)})$ follow immediately by examination of the proof of Theorem 5.1 and the restriction $k \leq ch^2$ still applies. These rates are almost optimal (modulo ε) in view of Theorem 2.2 (for $1 < p < \infty$, the regularity of the solution is $H_p^{1/2,\,1/4}(Q)$).

CHAPTER VI
PROOFS OF THEOREMS 3.1, 4.2, 5.1, 5.2, AND LEMMA 4.1

6.1 Opening Remarks

In this chapter we shall present the proofs of the main theorems which have been stated in this work. Each of these results states the approximation properties of the operators corresponding to one of the schemes we have defined. Except for Theorem 3.1, which was established in Lasiecka [12], all the theorems proved in this chapter are original. Also, the proofs contained in this chapter are all in the context of the abstract problem (2.6).

6.2 Proof of Theorem 3.1

We recall (2.37) and (3.11).

$$T^*v(t) = B^* \int_t^T S^*(\tau-t)v(\tau)dt ,$$

$$T_h^* v(t) = B^* \int_t^T S_h^*(\tau-t)P_h v(\tau)d\tau .$$

For the purpose of estimating $(T^*-T_h^*)v(t)$, we let

$$(T^* - T_h^*)v(t) = e_1^*(t) + e_2^*(t)$$

where

$$e_1^*(t) = B^*(I-P_h) \int_t^T S^*(\tau-t)v(\tau)dt$$

and

$$e_2^*(t) = B^* \int_t^T [P_h S^*(\tau-t) - S_h^*(\tau-t)P_h]v(\tau)dt.$$

We first condsider the case $p = \infty$. For $t \leq T-h^4$, we have

$$\left|e_1^*(t)\right| \leq \left|B^*(I-P_h) \int_t^{t+h^4} S^*(\tau-t)v(\tau)d\tau\right|$$

$$+ \left|B^*(I-P_h) \int_{t+h^4}^T S^*(\tau-t)v(\tau)d\tau\right|.$$

By applying (3.3) to the first term with $\alpha = 1/4 - 2\varepsilon$, and (3.2) to the second term, we get

$$\left|e_1^*(t)\right| \leq ch^{2(1-q-\varepsilon)} \left| \int_t^{t+h^4} A^{*1-\varepsilon} S^*(\tau-t)v(\tau)d\tau\right|_H$$

$$+ ch^{2(1-\bar{q})} \left| \int_{t+h^4}^T A^* S^*(\tau-t)v(\tau)d\tau\right|_H.$$

Then, by analyticity of the semigroup $S^*(t)$,

$$\left|e_1^*(t)\right|_U \leq ch^{2(1-q-\varepsilon)} \int_t^{t+h^4} \frac{1}{(\tau-t)^{1-\varepsilon}} d\tau \, |v|_{L_\infty[0,T;H]}$$

$$+ ch^{2(1-\bar{q})} \int_{t+h^4}^T \frac{1}{(\tau-t)} d\tau \, |v|_{L_\infty[0,T;H]}$$

$$\leq ch^{2(1-q-\varepsilon)}h^{4\varepsilon}|v|_{L_\infty[0,T;H]} + ch^{2(1-\bar{q})}\ \ln\frac{1}{h^4}|v|_{L_\infty[0,T;H]}$$

$$\leq c(h^{2(1-q+\varepsilon)} + h^{2(1-\bar{q})}\ln\frac{1}{h})|v|_{L_\infty[0,T;H]}$$

Similarly, for $t \geq T-h^4$, we have by application of (3.3) with $\alpha = 1/4 - 2\varepsilon$,

$$|e_1^*(t)|_U = |B^*(I-P_h)\int_t^T S^*(\tau-t)v(\tau)d\tau|_U$$

$$\leq ch^{2(1-q-\varepsilon)}|\int_t^T A^{*1-\varepsilon}S^*(\tau-t)v(\tau)d\tau|_H$$

$$\leq ch^{2(1-q-\varepsilon)}\int_t^T \frac{1}{(\tau-t)^{1-\varepsilon}}\,d\tau|v|_{L_\infty[0,T;H]}$$

$$\leq ch^{2(1-q-\varepsilon)}(T-t)^{\varepsilon}|v|_{L_\infty[0,T;H]}$$

$$\leq ch^{2(1-q+\varepsilon)}|v|_{L_\infty[0,T;H]}$$

Thus we have shown that

$$|e_1^*|_{L_\infty[0,T;U]} \leq c(h^{2(1-q+\varepsilon)} + h^{2(1-\bar{q})}\ln\frac{1}{h})|v|_{L_\infty[0,T;H]}$$

We now consider the term $e_2^*(t)$. By (3.4), we have

$$|e_2^*(t)| = |B^*\int_t^T(P_hS^*(\tau-t)-S_h^*(\tau-t)P_h)v(\tau)d\tau|_U$$

$$\leq ch^{-2\bar{q}}\int_t^T |P_hS^*(\tau-t)-S_h^*(\tau-t)P_h|_{H\to H}\ dt|v|_{L_\infty[0,T;H]}$$

In order to estimate the integrand, we shall need to use a result derived in Lasiecka [11].

Proposition 6.1 [11]. With A, A^* satisfying (3.5), (3.6), we have

$$|S_h(t)P_h - P_hS(t)|_{H\to H} \le c\,\frac{h^{2\ell}}{t^{\ell}}\,, \qquad 0 \le \ell \le 1.$$

//

For $t \le T-h^2$, we have

$$|e_2^*(t)|_U \le ch^{-2\bar{q}} \int_t^{t+h^2} |P_hS^*(\tau-t)-S_h^*(\tau-t)P_h|_{H\to H}\,d\tau\,|v|_{L_\infty[0,T;H]}$$

$$+\; ch^{-2\bar{q}} \int_{t+h^2}^{T} |P_hS^*(\tau-t)-S_h^*(\tau-t)P_h|_{H\to H}\,d\tau\,|v|_{L_\infty[0,T;H]}$$

By applying Proposition 6.1 with $\ell = 1-\varepsilon$ to the first integral and $\ell = 1$ to the second integral, we obtain

$$|e_2^*(t)|_U \le ch^{-2\bar{q}}\,[\int_t^{t+h^2} \frac{h^{2-2\varepsilon}}{(\tau-t)^{1-\varepsilon}}\,d\tau + \int_{t+h^2}^{T} \frac{h^2}{(\tau-t)}\,d\tau\,]\,|v|_{L_\infty[0,T;H]}$$

$$\le c(h^{2(1-\bar{q}-\varepsilon)}h^{2\varepsilon} + h^{2(1-\bar{q})}\ln(\frac{T-t}{h^2}))\,|v|_{L_\infty[0,T;H]}$$

$$\le c(h^{2(1-\bar{q})} + h^{2(1-\bar{q})}\ln\frac{1}{h})\,|v|_{L_\infty[0,T;H]}.$$

For $t \ge T-h^2$, we also apply Proposition 6.1 with $\ell = 1-\varepsilon$ to get

$$\|e_2^*(t)\|_U \le ch^{-2\bar{q}} \int_t^T \frac{h^{2-2\varepsilon}}{(\tau-t)^{1-\varepsilon}}\,d\tau\,\|v\|_{L_\infty[0,T;H]}$$

$$\le ch^{2(1-\bar{q}-\varepsilon)}(T-t)^{\varepsilon}\|v\|_{L_\infty[0,T;H]}$$

$$\le ch^{2(1-\bar{q})}\|v\|_{L_\infty[0,T;H]}$$

Thus we have shown that

$$\|e_2^*\|_{L_\infty[0,T;U]} \le c(h^{2(1-\bar{q})} + h^{2(1-\bar{q})}\ln \frac{1}{h})\|v\|_{L_\infty[0,T;H]}$$

and this completes the proof of Theorem 3.1 for $p = \infty$.

As for the case $p = 1$,

$$\int_0^T \|e_1^*(t)\|_U dt = \int_0^T \|B^*(I-P_h)\int_t^T S^*(\tau-t)v(\tau)d\tau\|_U dt$$

$$\le \int_0^T \int_t^T \|B^*(I-P_h)S^*(\tau-t)v(\tau)\|_U d\tau dt$$

By reversing the order of integration,

$$\int_0^T \|e_1^*(t)\|_U dt \le \int_0^T \int_0^\tau \|B^*(I-P_h)S^*(\tau-t)v(\tau)\|_U dt d\tau$$

$$\le \int_0^{h^4}\int_0^{\tau} |B^*(I-P_h)S^*(\tau-t)v(\tau)|_U dtd\tau$$

$$+ \int_{h^4}^{T}\int_0^{\tau-h^4} |B^*(I-P_h)S^*(\tau-t)v(\tau)|_U dtd\tau$$

$$+ \int_{h^4}^{T}\int_{\tau-h^4}^{\tau} |B^*(I-P_h)S^*(\tau-t)v(\tau)|_U dtd\tau$$

We note that in order to estimate the first integral we shall use the fact that $0 \le \tau \le h^4$. Also the bound $\tau-h^4$ in the second and third integrals is well-defined since $h^4 \le \tau \le T$. By applying (3.3) with $\alpha = 1/4 - 2\varepsilon$ to the first and third integrals, and (3.2) to the second integral, we have

$$\int_0^{T} |e_1^*(t)|_U dt \le ch^{2(1-q-\varepsilon)} \int_0^{h^4}\int_0^{\tau} |A^{*1-\varepsilon}S^*(\tau-t)v(\tau)|_H dtd\tau$$

$$+ ch^{2(1-\bar{q})}\int_{h^4}^{T}\int_0^{\tau-h^4} |A^*S^*(\tau-t)v(\tau)|_H dtd\tau$$

$$+ ch^{2(1-q-\varepsilon)}\int_{h^4}^{T}\int_{\tau-h^4}^{\tau} |A^{*1-\varepsilon}S^*(\tau-t)v(\tau)|_H dtd\tau$$

From the analyticity of $S^*(t)$, it then follows that

$$\int_0^T \|e_1^*(t)\|_U dt \le ch^{2(1-q-\varepsilon)} \int_0^{h^4} \left(\int_0^{\tau} \frac{dt}{(\tau-t)^{1-\varepsilon}}\right) \|v(\tau)\|_H d\tau$$

$$+ ch^{2(1-\bar{q})} \int_{h^4}^{T} \left(\int_0^{\tau-h^4} \frac{dt}{\tau-t}\right) \|v(\tau)\|_H d\tau$$

$$+ ch^{2(1-q-\varepsilon)} \int_{h^4}^{T} \left(\int_{\tau-h^4}^{\tau} \frac{dt}{(\tau-t)^{1-\varepsilon}}\right) \|v(\tau)\|_H d\tau$$

Hence

$$\int_0^T \|e_1^*(t)\|_U dt \le ch^{2(1-q-\varepsilon)} \int_0^{h^4} \tau^{\varepsilon} \|v(\tau)\|_H d\tau$$

$$+ ch^{2(1-\bar{q})} \int_{h^4}^{T} \ln\left(\frac{\tau}{h^4}\right) \|v(\tau)\|_H dt$$

$$+ ch^{2(1-q-\varepsilon)} \int_{h^4}^{T} h^{4\varepsilon} \|v(\tau)\|_H d\tau$$

$$\le c\left(h^{2(1-q+\varepsilon)} + h^{2(1-\bar{q})} \ln \frac{1}{h}\right) \|v\|_{L_1[0,T;H]}$$

We now estimate $\|e_2^*\|_{L_1[0,T;U]}$. By (3.4),

$$\int_0^T \|e_2^*(\tau)\|_U dt = \int_0^T \left\|B^* \int_t^T (P_h S^*(\tau-t) - S_h^*(\tau-t)P_h) v(\tau) d\tau\right\|_U dt$$

$$\le ch^{-2\bar{q}} \int_0^T \int_t^T \|(P_h S^*(\tau-t) - S_h^*(\tau-t)P_h) v(\tau)\|_H d\tau dt$$

By reversing the order of integration,

$$\int_0^T \|e_2^*(t)\|_U dt \le ch^{-2\bar{q}} \int_0^T \int_0^\tau \|P_h S^*(\tau-t) - S_h^*(\tau-t)P_h)v(\tau)\|_H dt d\tau$$

$$\le ch^{-2\bar{q}} \int_0^{h^2} \int_0^\tau \|(P_h S^*(\tau-t) - S_h^*(\tau-t)P_h)v(\tau)\|_H dt d\tau$$

$$+ ch^{-2\bar{q}} \int_{h^2}^T \int_0^{\tau-h^2} \|(P_h S^*(\tau-t) - S_h^*(\tau-t)P_h)v(\tau)\|_H dt d\tau$$

$$+ ch^{-2\bar{q}} \int_{h^2}^T \int_{\tau-h^2}^\tau \|(P_h S^*(\tau-t) - S_h^*(\tau-t)P_h)v(\tau)\|_H dt d\tau$$

By applying Proposition 6.1 with $\ell = 1-\varepsilon$ to the first and third integrals, and $\ell = 1$ to the second integral, we obtain

$$\int_0^T \|e_2^*(t)\|_U dt \le ch^{-2\bar{q}} \int_0^{h^2} \int_0^\tau \frac{h^{2-2\varepsilon}}{(\tau-t)^{1-\varepsilon}}\, dt)\|v(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}} \int_{h^2}^T (\int_0^{\tau-h^2} \frac{h^2}{\tau-t}\, dt)\|v(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}} \int_{h^2}^T (\int_{\tau-h^2}^\tau \frac{h^{2-2\varepsilon}}{(\tau-t)^{1-\varepsilon}}\, dt)\|v(\tau)\|_H d\tau$$

$$\le ch^{2(1-\bar{q}-\varepsilon)} \int_0^{h^2} \tau^\varepsilon \|v(\tau)\|_H d\tau + ch^{2(1-\bar{q})} \int_{h^2}^T \ln(\frac{\tau}{h^2})\|v(\tau)\|_H d\tau$$

$$+ ch^{2(1-\bar{q}-\varepsilon)} \int_{h^2}^T h^{2\varepsilon}\|v(\tau)\|_H d\tau$$

$$\leq c(h^{2(1-\overline{q})} + h^{2(1-\overline{q})}\ln \frac{1}{h})\|v\|_{L_1[0,T;H]}$$

Since $(T^* - T_h^*)v(t) = e_1^*(t) + e_2^*(t)$, this concludes the proof of Theorem 3.1.

6.3 Proof of Lemma 4.1

We shall make use of an integral representation for $T(t)$ and E_k^n (see Pazy [18, pp. 32-33]).

Let $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$\Gamma_1 = \{re^{i\phi}, 0 \leq r < \infty\}$ and $\Gamma_2 = \{re^{-i\phi}, 0 \leq r < \infty\}$, with ϕ as stated in Lemma 4.1. Then

$$(6.3.1) \quad T(t) = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}R(\lambda,A)d\lambda$$

and

$$(6.3.2) \quad E_k^n = ((I-kA)^{-1})^n = \frac{1}{2\pi i}\int_\Gamma \frac{1}{(1-k\lambda)^n} R(\lambda,A)d\lambda$$

In order to establish (i) we shall need to prove the following technical lemma.

Lemma 6.2. Let $z = x+iy$ be given such that $|z| = 1$, $x < 0$. Then

(a) $\left|e^{n\tau z} - \frac{1}{(1-\tau z)^n}\right| \leq c_x n\tau^2 e^{\frac{n\tau x}{2}}$ for $0 \leq \tau \leq \frac{1}{|x|}$, $n \geq 1$

(b) $\frac{1}{|1-\tau z|^n} \leq e^{-\frac{n}{2}}$ for $\tau \geq \frac{1}{|x|}$, $n \geq 1$.

The constant c_x depends on x but not on n or τ.

//

Proof of Lemma 6.2. To prove (a) we first need to prove the following inequality

$$(6.3.3)\quad \frac{1}{|1-\tau z|} \leq |e^{\frac{\tau z}{2}}| \qquad \text{for } 0 \leq \tau \leq \frac{1}{|x|}.$$

Since $|1-\tau z| = \sqrt{1+\tau^2-2\tau x}$ and $|e^{\frac{\tau z}{2}}| = e^{\frac{\tau x}{2}}$, it is equivalent to show

$$(6.3.4)\quad e^{\tau x}(1+\tau^2 - 2\tau x) \geq 1 \qquad \text{for } 0 \leq \tau \leq \frac{1}{|x|} .$$

It is clear that (6.3.4) is valid for $\tau = 0$. It is therefore sufficient to show that the derivative in τ of $e^{\tau x}(1+\tau^2-2\tau x)$ is positive on the interval $[0, 1/|x|]$. We have

$$\frac{d}{d\tau} (e^{\tau x}(1+\tau^2-2\tau x)) = e^{\tau x}(x(1+\tau^2-2\tau x) + 2\tau-2x)$$

It is enough to study the sign of

$$(6.3.5)\quad x(1+\tau^2-2\tau x) + 2\tau-2x = x\tau^2 + 2(1-x^2)\tau - x.$$

As a quadratic polynomial in τ, the roots of (6.5) are

$$\frac{-(1-x^2) + \sqrt{(1-x^2)^2+x^2}}{x} .$$

At this point we make the following observations:

(i) Since $-1 \leq x < 0$, the root

$$\frac{-(1-x^2)+\sqrt{(1-x^2)^2+x^2}}{x} = \frac{1-x^2}{x}\left[-1 + \sqrt{1+\frac{x^2}{(1-x^2)^2}}\right]$$

is clearly negative.

(ii) the coefficient of τ is negative, which makes the graph (a parabola) concave downward.

(iii) the other root satisfies

$$\frac{-(1-x^2)-\sqrt{(1-x^2)^2+x^2}}{x} = -\frac{1}{x}\left[1-x^2+\sqrt{(1-x^2)^2+x^2}\right]$$

$$\geq \frac{1}{|x|}(1-x^2+|x|)$$

$$\geq \frac{1}{|x|}$$

from which (6.3.3) follows.

//

The next step in proving (a) is to show

(6.3.6) $$\left|e^{\tau z} - \frac{1}{1-\tau z}\right| \leq c_x \tau^2 \qquad \text{for } 0 \leq \tau \leq \frac{1}{|x|}.$$

First of all, we note that

$$\left|e^{\tau z} - \frac{1}{1-\tau z}\right| = \frac{|e^{\tau z}(1-\tau z)-1|}{|1-\tau z|}.$$

Since $|1-\tau z| = \sqrt{1+\tau^2-2\tau x} \geq 1$, it is enough to show

$$(6.3.7) \qquad |e^{\tau z}(1-\tau z)-1| \leq c_x\tau^2 \qquad \text{for } 0 \leq \tau \leq \frac{1}{|x|}.$$

Using a power series expansion for $e^{\tau z}$, we have

$$\begin{aligned} e^{\tau z}(1-\tau z) - 1 &= e^{\tau z} - \tau z e^{\tau z} - 1 \\ &= 1+\tau z + \frac{\tau^2 z^2}{2!} + \frac{\tau^3 z^3}{3!} + \ldots + \frac{\tau^n z^n}{n!} + \ldots \\ &\quad - 1 - \tau z - \tau z(\tau z) - \tau z \frac{\tau^2 z^2}{2!} - \ldots - \tau z \frac{\tau^{n-1} z^{n-1}}{(n-1)!} - \ldots \\ &= \sum_{k=2}^{\infty} \tau^k z^k \left(\frac{1}{k!} - \frac{1}{(k-1)!}\right) \\ &= \sum_{k=2}^{\infty} \tau^k z^k \left(- \frac{k-1}{k!}\right). \end{aligned}$$

In modulus,

$$\begin{aligned} |e^{\tau z}(1-\tau z)-1| &\leq \sum_{k=2}^{\infty} \tau^k \frac{k-1}{k!} \\ &\leq \tau \sum_{k=1}^{\infty} \frac{\tau^k}{k!} \\ &\leq \tau(e^{\tau}-1). \end{aligned}$$

We have

$$e^{\tau}-1 \leq c_x\tau \qquad \text{for } 0 \leq \tau \leq \frac{1}{|x|}$$

where of course c_x depends on x.

Thus (6.3.6) follows.

We are now in a position to conclude the proof of (a). For $n \geq 1$, we have

$$\left|e^{n\tau z} - \frac{1}{(1-\tau z)^n}\right| = \left|e^{\tau z} - \frac{1}{1-\tau z}\right| \left|\sum_{j=0}^{n-1} \frac{1}{(1-\tau z)^j} e^{(n-1-j)\tau z}\right|$$

$$\leq \left|e^{\tau z} - \frac{1}{1-\tau z}\right| \sum_{j=0}^{n-1} \left|\frac{1}{(1-\tau z)^j}\right| \left|e^{(n-1-j)\tau z}\right|$$

Since $\left|\frac{1}{(1-\tau z)^j}\right| = \frac{1}{|1-\tau z|^j}$ and by applying (6.3.3) and (6.3.6),

$$\left|e^{n\tau t} - \frac{1}{(1-\tau z)^n}\right| \leq c_x \tau^2 \sum_{j=0}^{n-1} \left|e^{\frac{\tau z}{2}}\right|^j \left|e^{(n-1-j)\tau z}\right|$$

$$\leq c_x \tau^2 \sum_{j=0}^{n-1} e^{\frac{j\tau x}{2}} e^{(n-1-j)\tau x}$$

$$\leq c_x \tau^2 \sum_{j=0}^{n-1} e^{(n-1)\tau x} e^{\frac{-j\tau x}{2}} .$$

For $0 \leq j \leq n-1$, and since $x < 0$, we have

$$e^{-\frac{j\tau x}{2}} \leq e^{-\frac{n-1}{2}\tau x} .$$

So

$$c_x\tau^2 \sum_{j=0}^{n-1} e^{(n-1)\tau x}\, e^{-\frac{j\tau x}{2}} \le c_x\tau^2 \sum_{j=0}^{n-1} e^{(n-1-\frac{n-1}{2})\tau x}$$

$$\le c_x\tau^2 n e^{\frac{n-1}{2}\tau x}$$

$$\le c_x\tau^2 n\, e^{\frac{n\tau x}{2}}\, e^{-\frac{\tau x}{2}}$$

$$\le c_x n\tau^2 e^{\frac{n\tau x}{2}} \qquad \text{for } 0 \le \tau \le \frac{1}{|x|} .$$

This concludes the proof of (a). As for (b), we have

$$\frac{1}{|1-\tau z|^n} = \frac{1}{(1+\tau^2-2\tau x)^{n/2}} = \frac{e^{-n/2}}{((1+\tau^2-2\tau x)e^{-1})^{n/2}}, \quad n = 1,2,\ldots$$

It is therefore sufficient to show that

$$(6.3.8) \quad 1 + \tau^2 - 2\tau x \ge e \qquad \text{for } \tau \ge \frac{1}{|x|} .$$

We now make the following observations

(i) $x < |x| \le 1 \le \frac{1}{|x|}$

(ii) The graph of $y = 1 + \tau^2 - 2\tau x$ is concave upward and has a minimum at $\tau = x$.

(iii) It follows from (ii) that $1+\tau^2-2\tau x$ is increasing for $\tau \geq x$, and thus by (i), also for $\tau \geq \frac{1}{|x|}$.

(iv) (6.3.8) is valid for $\tau = \frac{1}{|x|}$ since $1 + \frac{1}{x^2} + 2 \geq 4 \geq e$ from which (b) follows.

//

We now return to the proof of Lemma 4.1.
From (6.3.1) and (6.3.2), we have

$$T(nk) - E_k^n = \frac{1}{2\pi i} \int_\Gamma (e^{nk\lambda} - \frac{1}{(1-k\lambda)^n}) R(\lambda,A)d\lambda$$

where, as we recall, $\Gamma = \Gamma_1 \cup \Gamma_2$ with

$$\Gamma_1 = \{re^{i\phi},\ 0 \leq r < \infty\} \text{ and } \Gamma_2 = \{re^{-i\phi},\ 0 \leq r < \infty\}$$

and with ϕ as given in the statement of Lemma 4.1. Thus if we let $z = e^{i\phi} = x+iy$ and $\lambda = rz$, then

$$T(nk) - E_k^n = \frac{1}{2\pi i} \int_0^\infty (e^{nkrz} - \frac{1}{(1-krz)^n}) R(rz,A)zdr$$
$$+ \frac{1}{2\pi i} \int_0^\infty (e^{nkr\bar{z}} - \frac{1}{(1-kr\bar{z})^n}) R(r\bar{z},A)\bar{z}dr$$

From the analyticity of $T(t)$, we can apply (2.3). Hence,

$$|T(nk)-E_k^n| \le c \int_0^\infty \left|e^{nkrz} - \frac{1}{(1-krz)^n}\right| \frac{1}{1+|rz|}\,dr$$

$$+ c \int_0^\infty \left|e^{nkr\bar{z}} - \frac{1}{(1-kr\bar{z})^n}\right| \frac{1}{1+|r\bar{z}|}\,dr$$

$$\le c \int_0^{\frac{1}{k|x|}} \left[\left|e^{nkrz} - \frac{1}{(1-krz)^n}\right| + \left|e^{nkr\bar{z}} - \frac{1}{(1-kr\bar{z})^n}\right|\right] \frac{1}{1+r}\,dr$$

$$+ c \int_{\frac{1}{k|x|}}^\infty \left[\left|e^{nkrz} - \frac{1}{(1-krz)^n}\right| + \left|e^{nkr\bar{z}} - \frac{1}{(1-kr\bar{z})^n}\right|\right] \frac{1}{1+r}\,dr$$

The next step will require the following inequality:

$$(6.3.9) \qquad \frac{1}{|1-\tau z|^n} = \frac{1}{|1-\tau z|^{n-\varepsilon}} \cdot \frac{1}{|1-\tau z|^{\varepsilon}} \le e^{-\frac{n-\varepsilon}{2}} \cdot \frac{1}{\tau^{\varepsilon}}$$

for $\tau \ge \frac{1}{|x|}$, $n \ge 1$, and $\varepsilon > 0$

which can be readily seen since $|1-\tau z| \ge |\tau z| = \tau$. Therefore, by using Lemma 6.2 with $\tau = kr$, we have

$$|T(nk)-E_k^n| \le c \int_0^{\frac{1}{k|x|}} nk^2r^2 e^{\frac{nkrx}{2}} \frac{1}{r}\,dr + c \int_{\frac{1}{k|x|}}^\infty e^{nkrx} \frac{1}{r}\,dr$$

$$+ c \int_{\frac{1}{k|x|}}^\infty e^{-\frac{(n-\varepsilon)}{2}} \frac{1}{k^{\varepsilon} r^{\varepsilon}}\,dr$$

$$(6.3.10) \qquad \leq cnk^2 \int_0^{\frac{1}{k|x|}} r e^{\frac{nkrx}{2}} dr$$

$$+ c \int_{\frac{1}{k|x|}}^{\infty} e^{nkrx} \frac{1}{r} dr + ce^{-\frac{n-\varepsilon}{2}} k^{-\varepsilon} \int_{\frac{1}{k|x|}}^{\infty} \frac{1}{r^{1+\varepsilon}} dr.$$

If we set $s = -krx = kr|x|$, then for the first integral in (6.3.10), we have

$$cnk^2 \int_0^{\frac{1}{k|x|}} r e^{\frac{nkrx}{2}} dr = cnk^2 \int_0^1 \frac{s}{k|x|} e^{-\frac{ns}{2}} \frac{1}{k|x|} ds$$

$$\leq cn \int_0^1 s e^{-\frac{ns}{2}} ds$$

$$\leq cn(- \frac{2s}{n} e^{-\frac{ns}{2}} \Big|_0^1 + \frac{2}{n} \int_0^1 e^{-\frac{ns}{2}} ds)$$

$$\leq cn(- \frac{2}{n} e^{-\frac{n}{2}} + \frac{2}{n}(- \frac{2}{n}(e^{-\frac{n}{2}} - 1)))$$

$$\leq \frac{c}{n}(-2ne^{-\frac{n}{2}} - 4(e^{-\frac{n}{2}} -1))$$

$$\leq \frac{c}{n} \qquad\qquad , n \geq 1.$$

Similarly, for the second integral in (6.3.10), we have

$$c\int_{\frac{1}{k|x|}}^{\infty} e^{nkrx}\,\frac{1}{r}\,dr = c\int_{1}^{\infty} \frac{k|x|}{s}\,e^{-ns}\,\frac{1}{k|x|}\,ds$$

$$\leq c\int_{1}^{\infty} \frac{1}{s}\,e^{-ns}ds$$

$$\leq c\int_{1}^{\infty} \frac{1}{s}\cdot\frac{1}{ns}\,ds$$

$$\leq \frac{c}{n}$$

Finally, for the third integral in (6.3.10),

$$ce^{-\frac{n-\varepsilon}{2}}\,k^{-\varepsilon}\int_{\frac{1}{k|x|}}^{\infty} \frac{1}{r^{1+\varepsilon}}\,dr = ce^{-\frac{n-\varepsilon}{2}}\,k^{-\varepsilon}\int_{1}^{\infty} \frac{(k|x|)^{1+\varepsilon}}{s^{1+\varepsilon}}\cdot\frac{1}{k|x|}\,ds$$

$$\leq ce^{-\frac{n-\varepsilon}{2}}\int_{1}^{\infty} \frac{1}{s^{1+\varepsilon}}\,ds$$

$$\leq ce^{-\frac{n}{2}}$$

$$\leq \frac{c}{n}.$$

This concludes part (i) of Lemma 4.1.

In order to prove part (ii), we shall consider separately AE_k^n and $AT(nk)$.

First of all, it is clear from the analyticity of T(t) that

$$(6.3.11)\quad \|AT(nk)\|_{H\to H} \le \frac{c}{nk},\ n \ge 1.$$

For AE_k^n, we shall consider the cases n = 1 and n ≥ 2. For n = 1, we have

$$I = (1-kA)(1-kA)^{-1} = E_k - kAE_k,$$

so that

$$kAE_k = E_k - I \in \mathcal{L}(H \to H)$$

and thus we have established (ii) for n=1.

For n ≥ 2, we make use of the integral representation for AE_k^n, i.e.,

$$AE_k^n = \frac{1}{2\pi i}\int_\Gamma \lambda \frac{1}{(1-k\lambda)^n} R(\lambda,A)d\lambda$$

Similarly as in part (i), we let $z = e^{i\phi} = x+iy$ and $\lambda = rz$. Then

$$AE_k^n = \frac{1}{2\pi i}\int_0^\infty rz \frac{1}{(1-krz)^n} R(rz,A)zdr + \frac{1}{2\pi i}\int_0^\infty r\bar{z} \frac{1}{(1-kr\bar{z})^n} R(r\bar{z},A)\bar{z}dr.$$

Hence

$$\|AE_k^n\| \le c\int_0^\infty [\frac{r}{|1-krz|^n} \|R(rz,A)\| + \frac{r}{|1-kr\bar{z}|^n} \|R(r\bar{z},A)\|]dr$$

Again by analyticity of the semigroup generated by A, and by application of the inequality (6.3.9), we deduce that

$$
\begin{aligned}
|AE_k^n| &\le c\int_0^{\frac{1}{k|x|}} e^{\frac{nkrx}{2}}\,dr + c\int^{\infty} \frac{1}{(kr)^{1+\varepsilon}} e^{-\frac{n-1-\varepsilon}{2}}\,dr \\
&\le \frac{c}{nk}\left(e^{-\frac{nk}{2}}-1\right) + ce^{-\frac{n+1-\varepsilon}{k|x|}} k^{-1-\varepsilon}\int_{\frac{1}{k|x|}}^{\infty} \frac{1}{r^{1+\varepsilon}}\,dr \\
&\le \frac{c}{nk} \qquad \text{for } n \ge 2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|A(E_k^n - T(nk))|_{H\to H} &\le |AE_k^n|_{H\to H} + |AT(nk)|_{H\to H} \\
&\le \frac{c}{nk}, \qquad n \ge 1,
\end{aligned}
$$

which concludes the proof of Lemma 4.1.

6.4 Proof of Theorem 4.2

We have the following decomposition for $(T^*-T^*_k)v(t)$, where $(j-1)k < t \le jk$.

$$
\begin{aligned}
(T^*-T^*_k)v(t) &= T^*v(t) - T^*(P_kv)(t) \\
&\quad + T^*(P_kv)(t) - T^*(P_kv)((j-1)k) \\
&\quad + T^*(P_kv)((j-1)k) - T^*_k(P_kv)(jk)
\end{aligned}
$$

Remark. The apparent inconsistency in the last line of the above expression ((j-1)k vs. jk) is purely a matter of notation. In fact, one can see from the definition of P_k (see (4.8)) that $T^*_k(P_kv)(jk)$ does depend on the values of v over the interval $((j-1)k,T]$.

//

We have the further decomposition

$$(T^*-T_k^*)v(t) = T^*(I-P_k)v(t)$$

$$+ B^*\Big[\int_t^T S^*(\tau-t)P_k v(\tau)d\tau - \int_{(j-1)k}^T S^*(\tau-(j-1)k)P_k v(\tau)d\tau\Big]$$

$$+ B^* \sum_{n=j}^N \Big[\int_{(n-1)k}^{nk} (S^*(\tau-(j-1)k) - S^*((n+1-j)k))P_k v(\tau)d\tau\Big]$$

$$+ B^* \sum_{n=j}^N \Big[\int_{(n-1)k}^{nk} S^*((n+1-j)k)P_k v(\tau)d\tau - kE_k^{*n+1-j}P_k v(nk)\Big]$$

At this point we observe that

$$\int_{(n-1)k}^{nk} S^*((n+1-j)k)P_k v(\tau)d\tau = S^*((n+1-j)k) \int_{(n-1)k}^{nk} P_k v(\tau)d\tau$$

$$= k\, S^*((n+1-j)k)P_k v(nk)$$

Therefore,

$$(T^*-T_k^*)v(t) = T^*(I-P_k)v(t)$$

$$- B^* \int_{(j-1)k}^{t} S^*(\tau-(j-1)k)P_k v(\tau)d\tau$$

$$- B^*(S^*(t-(j-1)k)-I) \int_t^T S^*(\tau-t)P_k v(\tau)d\tau$$

$$+ B^* \sum_{n=j}^N \int_{(n-1)k}^{nk} (S^*(\tau-(j-1)k) - S^*((n+1-j)k))P_k v(\tau)d\tau$$

$$+ B^* \sum_{n=j}^N k(S^*((n+1-j)k) - E_k^{*n+1-j})P_k v(nk)$$

Let $e_1^*(t) = T^*(I-P_k)v(t) = T^*A^{*\alpha}(I-P_k)A^{*-\alpha}v(t)$

Since $L \in \mathcal{L}(L_p[0,T;U] \to H_p^{1-q}[0,T;H])$ for $p = 1,\infty$ (see Theorem 2.2), it follows from Lemma 2.3 that

$$(6.4.1) \quad T^* \in \mathcal{L}(H_p^{-(1-q)}[0,T;H] \to L_p[0,T;U]) \text{ for } p = 1,\infty.$$

Also, we recall from section 4.2 that

$$\|(I-P_k)x\|_{L_p[0,T;H]} \leq ck^{\beta}\|x\|_{H_p^{\beta}[0,T;H]} \quad \text{for } p = 1,\ \infty,\ 0 \leq \beta \leq 1.$$

Again by Lemma 2.3 (and by selfadjointness of P_k), we have

$$(6.4.2) \quad \|(I-P_k)x\|_{H_p^{-\beta}[0,T;H]} \leq ck^{\beta}\|x\|_{L_p[0,T;H]} \quad \text{for } p = 1,\ \infty,\ 0 \leq \beta \leq 1.$$

By taking $\beta = 1-q-\alpha$ in (6.4.2), $x = A^{*-\alpha}v(t)$, and by combining (6.4.1) with (6.4.2), it follows that

$$\begin{aligned}\|T^*A^{*\alpha}(I-P_k)A^{*-\alpha}v\|_{L_p[0,T;U]} &= \|e_1^*\|_{L_p[0,T;U]} \\ &\leq ck^{1-q-\alpha}\|v\|_{L_p[0,T;D(A^{*-\alpha})]}\end{aligned}$$

for $p = 1,\ \infty,\ 0 \leq \alpha \leq 1-q$.

Since $A^{-q}B \in \mathcal{L}(U \to H)$, it follows from Lemma 2.3 that

$$(6.4.3) \quad B^*A^{*-q} \in \mathcal{L}(H \to U)$$

We shall be using this fact repeatedly throughout the remainder of this proof and in subsequent proofs.

Let $e_2^*(t) = B^* \int_{(j-1)k}^{t} S^*(\tau-(j-1)k)P_k v(\tau)d\tau$. By (6.4.3.),

$$\|e_2^*(t)|_U = |B^*A^{*-q} \int_{(j-1)k}^{t} A^{*q+\alpha}S^*(\tau-(j-1)k)A^{*-\alpha}P_k v(\tau)d\tau\|_U$$

$$\leq c \int_{(j-1)k}^{t} |A^{*q+\alpha}S^*(\tau-(j-1)k)A^{*-\alpha}P_k v(\tau)|_H d\tau$$

$$\leq c \int_{(j-1)k}^{jk} |A^{*q+\alpha}S^*(\tau-(j-1)k)A^{*-\alpha}P_k v(\tau)|_H d\tau$$

$$\leq c \int_{(j-1)k}^{jk} \frac{1}{(\tau-(j-1)k^{q+\alpha}} |A^{*-\alpha}P_k v(\tau)|d\tau$$

$$\leq c \left(\int_{(j-1)k}^{jk} \frac{1}{(\tau-(j-1)k^{q+\alpha}} d\tau\right) |A^{*-\alpha}P_k v(jk)|$$

$$\leq ck^{1-q-\alpha}|A^{*-\alpha}P_k v(jk)\|_H$$

For p = 1 we have

$$\int_0^T |e_2^*(t)|_U = \sum_{j=1}^{N} \int_{(j-1)k}^{jk} |e_2^*(t)|dt$$

$$\leq \sum_{j=1}^{N} k\cdot ck^{1-q-\alpha}|A^{*-\alpha}P_k v(jk)|_H$$

$$\leq ck^{1-q-\alpha}\cdot k \sum_{j=1}^{N} |P_k v(jk)|_{D(A^{*-\alpha})}$$

$$\leq ck^{1-q-\alpha} \sum_{j=1}^{N} \left|\int_{(j-1)k}^{jk} v(\tau)d\tau\right|_{D(A^{*-\alpha})}$$

$$\leq ck^{1-q-\alpha}|v|_{L_1[0,T;D(A^{*-\alpha})]}.$$

For p = ∞, we have similarly

$$\operatorname*{ess\,sup}_{0\leq t\leq T} |e_2^*(t)|_U = \max_{j=1,\dots,N} \{\operatorname*{ess\,sup}_{(j-1)k<t\leq jk} |e_2^*(t)|_U\}$$

$$\leq \max_{j=1,\dots,N} ck^{1-q-\alpha}|A^{*-\alpha}P_k v(jk)|_H$$

$$\leq ck^{1-q-\alpha} \max_{j=1,\dots,N} |P_k v(jk)|_{D(A^{*-\alpha})}$$

$$\leq ck^{1-q-\alpha}|v|_{L_\infty[0,T;D(A^{*-\alpha})]}$$

We now turn to the estimation of $e_3^*(t)$, where

$$e_3^*(t) = B^*(S^*(t-(j-1)k-I) \int_t^T S^*(\tau-t)P_k v(\tau)d\tau$$

Again by (6.4.3)

$$|e_3^*(t)|_U = |B^*A^{*-\alpha}A^{*q+\alpha}(S^*(t-(j-1)k)-I) \int_t^T S^*(\tau-t)A^{*-\alpha}P_k v(\tau)d\tau|_U$$

$$\leq c|A^{*q+\alpha}(S^*(t-(j-1)k)-I) \int_t^T S^*(\tau-t)A^{*-\alpha}P_k v(\tau)d\tau|_H$$

$$\leq c|A^{*q+\alpha} \int_0^{t-(j-1)k} A^*S^*(\xi)d\xi \int_t^T S^*(\tau-t)A^{*-\alpha}P_k v(\tau)d\tau|_H$$

$$\leq c \int_0^{t-(j-1)k} |A^{*q+\alpha+\varepsilon}S^*(\xi)|d\xi \int_t^T |A^{*1-\varepsilon}S^*(\tau-t)A^{*-\alpha}P_k v(\tau)|d\tau$$

$$\leq c \int_0^k \frac{1}{\xi^{q+\alpha+\varepsilon}}\, d\xi \int_t^T \frac{1}{(\tau-t)^{1-\varepsilon}}\, |A^{*-\alpha}P_k v(\tau)|d\tau.$$

$$\leq ck^{1-q-\alpha-\varepsilon} \int_t^T \frac{1}{(\tau-t)^{1-\varepsilon}}|A^{*-\alpha}P_k v(\tau)|d\tau$$

For $p = 1$, we have

$$\int_0^T |e_3^*(t)|_U dt \leq ck^{1-q-\alpha-\varepsilon} \int_0^T \int_t^T \frac{1}{(\tau-t)^{1-\varepsilon}}\, |A^{*-\alpha}P_k v(\tau)|d\tau dt$$

By reversing the order of integration,

$$\int_0^T \|e_3^*(t)\|_U dt \le ck^{1-q-\alpha-\varepsilon} \int_0^T \left(\int_0^\tau \frac{1}{(\tau-t)^{1-\varepsilon}}\, dt\right) \|A^{*-\alpha} P_k v(\tau)\| d\tau$$

$$\le ck^{1-q-\alpha-\varepsilon} \int_0^T \tau^\varepsilon \|A^{*-\alpha} P_k v(\tau)\| d\tau$$

$$\le cT^{\varepsilon/2}\, k^{1-q-\alpha-\varepsilon} \int_0^T \|P_k v(\tau)\|_{D(A^{*-\alpha})} d\tau$$

$$\le ck^{1-q-\alpha-\varepsilon} \|v\|_{L_1[0,T;D(A^{*-\alpha})]}$$

For p = ∞, we have similarly,

$$\operatorname*{ess\,sup}_{0\le t\le T} \|e_3^*(t)\|_U \le ck^{1-q-\alpha-\varepsilon} \sup_{0\le t\le T} \int_t^T \frac{1}{(\tau-t)^{1-\varepsilon}} \|A^{*-\alpha} P_k v(\tau)\| d\tau$$

$$\le ck^{1-q-\alpha-\varepsilon} \left(T^\varepsilon \int_0^T d\tau\right) \sup_{0\le\tau\le T} \|P_k v(\tau)\|_{D(A^{*-\alpha})}$$

$$\le ck^{1-q-\alpha-\varepsilon} \|v\|_{L_\infty[0,T;D(A^{*-\alpha})]}$$

Now we let

$$e_4^*(t) = B^* \sum_{n=j}^{N} \int_{(n-1)k}^{nk} (S^*(\tau-(j-1)k) - S^*((n+1-j)k)) P_k v(\tau) d\tau$$

As before, by (6.4.3),

$$|e_4^*(t)|_U = |B^*A^{*-q}A^{*q+\alpha}\sum_{n=j}^{N}\int_{(n-1)k}^{nk} (S^*(\tau-(j-1)k)-S^*((n+1-j)k))A^{*-\alpha}P_k v(\tau)d\tau|_U$$

$$\leq c|A^{*q+\alpha}\sum_{n=j}^{N}\int_{(n-1)k}^{nk} (S^*(\tau-(j-1)k)-S^*((n+1-j)k))A^{*-\alpha}P_k v(\tau)d\tau|_H$$

In order to estimate $|e_4^*(t)|$ we shall need to use interpolation. Let

$$a_j = \sum_{n=j}^{N}\int_{(n-1)k}^{nk} A^{*-\varepsilon}(S^*(\tau-(j-1)k)-S^*((n+1-j)k))A^{*-\alpha}P_k v(\tau)d\tau,$$

and furthermore, let $a(t) = a(jk) = a_j$ for $(j-1)k < t \leq jk$.

Then

$$|e_4^*(t)|_U \leq c|A^{*q+\alpha+\varepsilon}a(t)|_H,$$

so that

$$|e_4^*|_{L_p[0,T;U]} \leq c|a|_{L_p[0,T;D(A^{*q+\alpha+\varepsilon})]} \quad \text{for } p = 1, \infty.$$

It is clearly sufficient to estimate $|a|_{L_p[0,T;D(A^{*q+\alpha+\varepsilon})]}$, and we shall do so by estimating

$$|a|_{L_p[0,T;H]} \text{ and } |a|_{L_p[0,T;D(A^*)]}.$$

Then we shall make use of a standard interpolation theorem [12, Vol. I, p. 27] to conclude the argument.

We have

$$|a_j|_H = \Big| \sum_{n=j}^{N} \int_{(n-1)k}^{nk} A^{*-\varepsilon}(S^*(\tau(j-1)k)-S^*((n+1-j)k))A^{*-\alpha}P_k v(\tau)d\tau \Big|_H$$

$$\leq \sum_{n=j}^{N} \int_{(n-1)k}^{nk} |A^{*-\varepsilon}(S^*(\tau-(j-1)k)-S^*((n+1-j)k))A^{*-\alpha}P_k v(\tau)d\tau|_H d\tau$$

$$\leq \sum_{n=j}^{N} \int_{(n-1)k}^{nk} |A^{*-\varepsilon}(\int_{\tau-(j-1)k}^{(n+1-j)k} A^*S^*(\xi)d\xi)A^{*-\alpha}P_k v(\tau)|_H d\tau$$

$$\leq \sum_{n=j}^{N} \int_{(n-1)k}^{nk} (\int_{\tau-(j-1)k}^{(n+1-j)k} |A^{*1-\varepsilon}S^*(\xi)|d\xi)|A^{*-\alpha}P_k v(\tau)|_H d\tau$$

$$\leq \sum_{n=j}^{N} \int_{(n-1)k}^{nk} (\int_{(n-j)k}^{(n+1-j)k} \frac{c}{\xi^{1-\varepsilon}} d\xi)|A^{*-\alpha}P_k v(\tau)|_H d\tau$$

$$\leq \sum_{n=j}^{N} \frac{c}{\varepsilon}[((n+1-j)k)^{\varepsilon} - ((n-j)k)^{\varepsilon}] \int_{(n-1)k}^{nk} |A^{*-\alpha}P_k v(nk)|_H d\tau$$

$$\leq ck \sum_{n=j}^{N} [((n+1-j)k)^{\varepsilon} - ((n-j)k)^{\varepsilon}]|A^{*-\alpha}P_k v(nk)|_H$$

Therefore,

$$|a|_{L_1[0,T;H]} = k \sum_{j=1}^{N} |a_j|_H$$

$$\leq ck^2 \sum_{j=1}^{N} \sum_{n=j}^{N} [((n+1-j)k)^{\varepsilon} - ((n-j)k)^{\varepsilon}]|A^{*-\alpha}P_k v(nk)|_H$$

By reversing the order of summation,

$$
\begin{aligned}
|a|_{L_1[0,T;H]} &\le ck^2 \sum_{n=1}^{N} \Big(\sum_{j=1}^{n} [((n+1-j)k)^{\varepsilon}-((n-j)k)^{\varepsilon}]\Big) |A^{*-\alpha}P_k v(nk)|_H \\
&\le ck^2 \sum_{n=1}^{N} (nk)^{\varepsilon} |A^{*-\alpha}P_k v(nk)|_H \\
&\le ckT^{\varepsilon}\Big(k \sum_{n=1}^{N} |A^{*-\alpha}P_k v(nk)|_H\Big) \\
&\le ck|v|_{L_1[0,T;D(A^{*-\alpha})]}
\end{aligned}
$$

Similarly, for $p = \infty$,

$$
\begin{aligned}
|a|_{L_\infty[0,T;H]} &= \max_{j=1,\dots,N} |a_j|_H \\
&\le ck \max_{j=1,\dots,N} \sum_{n=j}^{N} [((n+1-j)k)^{\varepsilon}-((n-j)k)^{\varepsilon}] |A^{*-\alpha}P_k v(nk)|_H \\
&\le ck \max_{j=1,\dots,N} |A^{*-\alpha}P_k v(nk)|_H \sum_{n=1}^{N} [((n+1-j)k)^{\varepsilon}-((n-j)k)^{\varepsilon}] \\
&\le ck(T^{\varepsilon}) |P_k v|_{L_\infty[0,T;D(A^{*-\alpha})]} \\
&\le ck|v|_{L_\infty[0,T;D(A^{*-\alpha})]}
\end{aligned}
$$

The next step is to estimate $|a_j|_{D(A^*)}$. We have

$$\|a_j\|_{D(A^*)} = \|A^* a_j\|_H$$

$$= \Big\| \sum_{n=j}^{N} \int_{(n-1)k}^{nk} A^{*1-\varepsilon}(S^*(\tau-(j-1)k)-S^*((n+1-j)k))A^{*-\alpha}P_k v(\tau)d\tau \Big\|_H$$

$$\leq \sum_{n=j}^{N} \int_{(n-1)k}^{nk} \|A^{*1-\varepsilon}(S^*(\tau-(j-1)k)-S^*((n+1-j)k))A^{*-\alpha}P_k v(\tau)\|_H d\tau$$

We estimate this expression crudely by using the triangle inequality

$$\|a_j\|_{D(A^*)} \leq \sum_{n=j}^{N} \int_{(n-1)k} \|A^{*1-\varepsilon}S^*(\tau-(j-1)k\|_{H\to H}$$

$$+ \sum_{n=j}^{N} \int_{(n-1)k}^{nk} \|A^{*1-\varepsilon}S^*((n+1-j)k)\|_{H\to H}d\tau \|A^{*-\alpha}P_k v(nk)\|_H$$

$$\leq \sum_{n=j}^{N} \int_{(n-1)k}^{nk} \Big(\frac{c}{(\tau-(j-1)k)^{1-\varepsilon}} + \frac{c}{((n+1-j)k)^{1-\varepsilon}}\Big)d\tau \|A^{*-\alpha}P_k v(nk)\|_H$$

$$\leq c \sum_{n=j}^{N} [((n+1-j)k)^{\varepsilon} - ((n-j)k)^{\varepsilon} + \frac{k}{((n+1-j)k)^{1-\varepsilon}}] \|A^{*-\alpha}P_k v(nk)\|_H$$

For p = 1, we have

$$\|a\|_{L_1[0,T;D(A^*)]} = k \sum_{j=1}^{N} \|a_j\|_{D(A^*)}$$

$$\leq ck \sum_{j=1}^{N} \sum_{n=j}^{N} [((n+1-j)k)^{\varepsilon}-((n-j)k)^{\varepsilon}+ \frac{k}{((n+1-j)k)^{1-\varepsilon}}] \|A^{*-\alpha}P_k v(nk)\|_H$$

$$\leq ck \sum_{n=1}^{N} \Big(\sum_{j=1}^{n} [((n+1-j)k)^{\varepsilon} - ((n-j)k)^{\varepsilon} + \frac{k}{((n+1-j)k)^{1-\varepsilon}}]\Big) |A^{*-\alpha} P_k v(nk)|_H$$

$$\leq ck^{1+\varepsilon} \sum_{n=1}^{N} (n^{\varepsilon} + 1 + \int_1^n \frac{1}{x^{1-\varepsilon}} dx) |A^{*-\alpha} P_k v(nk)|_H$$

$$\leq c(Nk)^{\varepsilon} \sum_{n=1}^{N} k |P_k v(nk)|_{D(A^{*-\alpha})}$$

$$\leq c\, |v|_{L_1[0,T;D(A^{*-\alpha})]}$$

Similarly, for $p = \infty$,

$$|a|_{L_\infty[0,T;D(A^*)]} = \max_{j=1,\dots,N} |a_j|_{D(A^*)}$$

$$\leq c \max_{j=1,\dots,N} \sum_{n=j}^{N} [((n+1-j)\overset{\varepsilon}{k}) - ((n-j)\overset{\varepsilon}{k}) + \frac{k}{((n+1-j)k)^{1-\varepsilon}}] |A^{*-\alpha} P_k v(nk)|_H$$

$$\leq ck^{\varepsilon} \max_{j=1,\dots,N} |A^{*-\alpha} P_k v(jk)| \sum_{n=1}^{N} ((n+1-j)^{\varepsilon} - (nk)^{\varepsilon} + \frac{1}{(n+1-j)^{1-\varepsilon}})$$

$$\leq c(Nk)^{\varepsilon} |P_k v|_{L_\infty[0,T;D(A^{*-\alpha})]}$$

$$\leq c|v|_{L_\infty[0,T;D(A^{*-\alpha})]}$$

Let us summarize the results we have just obtained:

$$|a|_{L_p[0,T;H]} \leq ck|v|_{L_p[0,T;D(A^{*-\alpha})]} \quad , \ p = 1,\infty$$

$$|a|_{L_p[0,T;D(A^*)]} \leq c|v|_{L_p[0,T;D(A^{*-\alpha})]} \quad , \ p = 1,\ \infty.$$

By applying the interpolation theorem to each pair of estimates, we get

$$|a|_{L_p[0,T;D(A^{*\beta})]} \leq ck^{1-\beta}|v|_{L_p[0,T;D(A^{*-\alpha})]} \quad , \ p = 1,\infty,\ 0 \leq \beta \leq 1.$$

We are concerned with estimates in $L_p[0,T;D(A^{*q+\alpha+\varepsilon})]$, hence we take $\beta = q + \alpha + \varepsilon$ to obtain

$$|e_4^*|_{L_p[0,T;U]} \leq c\ |a|_{L_p[0,T;D(A^{*q+\alpha+\varepsilon})]} \leq ck^{1-q-\alpha-\varepsilon}|v|_{L_p[0,T;D(A^{*-\alpha})]}$$

for $p = 1,\ \infty$.

This completes the estimation of e_4^*.

Finally, we let

$$e_5^*(t) = kB^* \sum_{n=j}^{N} (S^*((n+1-j)k) - E_k^{*n+1-j})P_k v(nk)$$

$$= kB^*A^{*-q} \sum_{n=j}^{N} A^{*q+\alpha}(S^*((n+1-j)k) - E_k^{*n+1-j})A^{*-\alpha}P_k v(nk)$$

By using (6.4.3) as before,

$$\|e_5^*(t)\|_U \le ck \sum_{n=j}^{N} \|A^{*q+\alpha}(S^*((n+1-j)k)-E_k^{*n+1-j})A^{*-\alpha}P_k v(nk)\|_H$$

In order to derive an estimate for $\|e_5^*\|$, we shall use interpolation once again. We have by Lemma 4.1:

$$\|(S^*((n+1-j)k)-E_k^{*n+1-j})A^{*-\alpha}P_k v(nk)\|_H \le \frac{c}{n+1-j}\|A^{*-\alpha}P_k v(nk)\|_H$$

and also

$$\|A^*(S^*((n+1-j)k)-E_k^{*n+1-j})A^{*-\alpha}P_k v(nk)\|_H \le \frac{c}{(n+1-j)k}\|A^{*-\alpha}P_k v(nk)\|_H$$

By application of the interpolation theorem, we obtain

$$\|A^{*\beta}(S^*((n+1-j)k)-E_k^{*n+1-j})A^{*-\alpha}P_k v(nk)\|_H \le \frac{ck^{-\beta}}{n+1-j}\|A^{*-\alpha}P_k v(nk)\|_H$$

By taking $\beta = q+\alpha$, we get

$$\|e_5^*(t)\|_U \le ck \sum_{n=j}^{N} \frac{ck^{-q-\alpha}}{n+1-j}\|A^{*-\alpha}P_k v(nk)\|_H$$

$$\le ck^{1-q-\alpha-\varepsilon} \sum_{n=j}^{N} k^{\varepsilon}.\ \frac{1}{n+1-j}\|A^{*-\alpha}P_k v(nk)\|_H$$

For $p = 1$,

$$\int_0^T \|e_5^*(t)\|_U dt = k\sum_{j=1}^{N} \|e_5^*(jk)\|_U$$

$$\le ck\cdot k^{1-q-\alpha-\varepsilon} \sum_{j=1}^{N}\sum_{n=j}^{N} k^{\varepsilon}\frac{1}{n+1-j}\|A^{*-\alpha}P_k v(nk)\|_H$$

$$\leq ck\cdot k^{1-q-\alpha-\varepsilon}\sum_{n=1}^{N}\Big(\sum_{j=1}^{n}k^{\varepsilon}\frac{1}{n+1-j}\Big)\|A^{*-\alpha}P_k v(nk)\|_H$$

$$\leq c(k^{\varepsilon}\ln N)k^{1-q-\alpha-\varepsilon}\sum_{n=1}^{N}k\|P_k v(nk)\|_{D(A^{*-\alpha})}$$

$$\leq ck^{1-q-\alpha-\varepsilon}\|v\|_{L_1[0,T;D(A^{*-\alpha})]}.$$

For $p = \infty$, we have

$$\operatorname*{ess\,sup}_{0\leq t\leq T}\|e_5^*(t)\| = \max_{j=1,\dots,N}\|e_5^*(jk)\|_U$$

$$\leq ck^{1-q-\alpha-\varepsilon}\max_{j=1,\dots,N}\sum_{n=j}^{N}k^{\varepsilon}\frac{1}{n+1-j}\|A^{*-\alpha}P_k v(nk)\|_H$$

$$\leq ck^{1-q-\alpha-\varepsilon}(k^{\varepsilon}\ln N)\max_{j=1,\dots,N}\|A^{*-\alpha}P_k v(jk)\|_H$$

$$\leq ck^{1-q-\alpha-\varepsilon}\|v\|_{L_\infty[0,T;D(A^{*-\alpha})]}.$$

The proof of Theorem 4.2 is now completed, since

$$(T^*-T_k^*)v(t) = e_1^*(t) + e_2^*(t) + e_3^*(t) + e_4^*(t) + e_5^*(t).$$

6.5 Proof of Theorem 5.1

Let $t \in [0,T]$ and let j be the unique integer such that $(j-1)k < t \leq jk$. We recall that by definition

$$(6.5.1)\quad T_h^* v(t) = T_h^* P_h v(t)$$

and

$$(6.5.2)\quad T_{hk}^* v(t) = (T_{hk}^* P_h P_k v)(t) = (T_{hk}^* P_h P_k v)(jk) \text{ for } t \in [0,T].$$

We shall use the following decomposition for $(T^*-T_{hk}^*)v(t)$:

$$\begin{aligned}(T^*-T_{hk}^*)v(t) &= (T^*-T_h^*)v(t) + (T_h^*-T_{hk}^*)v(t)\\ &= (T^*-T_h^*)v(t) + T_h^* v(t) - T_h^* v((j-1)k)\\ &\qquad + T_h^* v((j-1)k) - T_{hk}^* v(jk).\end{aligned}$$

Remark. The apparent inconsistency in the expression just above (jk vs. (j-1)k) is purely a matter of notation. As it turns out, $T_{hk}^* v(jk)$ does depend on the values of v over the interval $((j-1)k,T]$ (see the definition of P_k in section 4.2).

//

In the course of the proof we will make crucial use of the inverse approximation property (3.4), which we recall here:

$$(6.5.3) \qquad |B^*v_h|_U \le ch^{-2\bar{q}}|v_h|_H \qquad \forall v_h \in V_h.$$

for some $\bar{q} \le q$.

Let $e_1^*(t) = (T^*-T_h^*)v(t)$. By Theorem 3.1, the following estimates hold

$$(6.5.4) \qquad |e_1^*|_{L_p[0,T;U]} \le ch^{2(1-\bar{q})}\ln\frac{1}{h}|v|_{L_p[0,T;H]} \qquad \text{for } p = 1,\infty.$$

Now let $e_2^*(t) = T_h^*v(t) - T_h^*v((j-1)k)$. As we have noted in (6.5.1), we can assume without loss of generality that $v(t) = P_hv(t) = v_h(t)$ for $t\ [0,T]$. Then

$$e_2^*(t) = B^*[\int_t^T S_h^*(\tau-t)v_h(\tau)d\tau - \int_{(j-1)k}^T S_h^*(\tau-(j-1)k)v_h(\tau)d\tau]$$

$$= -B^*(S_h^*(t-(j-1)k)-I)\int_t^T S_h^*(\tau-t)v_h(\tau)d\tau$$

$$-B^*\int_{(j-1)k}^t S_h^*(\tau-(j-1)k)v_h(\tau)d\tau$$

By (6.5.3) and by differentiability of $S_h^*(t)$, we have

$$|e_2^*(t)|_U \le |B^*(S_h^*(t-(j-1)k)-I)\int_t^T S_h^*(\tau-t)v_h(\tau)d\tau|_U$$

$$+ |B^*\int_{(j-1)k}^t S_h^*(\tau-(j-1)k)v_h(\tau)d\tau|_U$$

$$\le ch^{-2\bar{q}}|\int_0^{t-(j-1)k} A_h^*S_h^*(\xi)d\xi \int_t^T S_h^*(\tau-t)v_h(\tau)d\tau|_H$$

$$+\ ch^{-2\bar{q}}\Big\| \int_{(j-1)k}^{t} S_h^*(\tau-(j-1)k)v_h(\tau)d\tau\Big\|_H$$

Let $p = \infty$. We need to consider two cases.

(i) $t + k \leq T$ (ii) $t + k > T$.

In both cases, we shall make use of the commutativity of A_h^* and $S_h^*(t)$. We also fix α, with $0 < \alpha < \frac{1}{2}$. In case (i),

$$\|e_2^*(t)\|_U \leq ch^{-2\bar{q}}\Big\| \int_0^{t-(j-1)k} A_h^{*\alpha}S_h^*(\xi)d\xi \int_t^{t+k} A_h^{*1-\alpha}S_h^*(\tau-t)v_h(\tau)d\tau\Big\|_H$$

$$+\ ch^{-2\bar{q}}\Big\| \int_0^{t-(j-1)k} S_h^*(\xi)d\xi \int_{t+k}^{T} A_h^*S_h^*(\tau-t)v_h(\tau)d\tau\Big\|_H$$

$$+\ ch^{-2\bar{q}}\Big\| \int_{(j-1)k}^{t} S_h^*(\tau-(j-1)k)v_h(\tau)d\tau\Big\|_H$$

$$\leq ch^{-2\bar{q}} \int_0^{k} \|A_h^{*\alpha}S_h^*(\xi)\|_{H\to H}d\xi \int_t^{t+k} \|A_h^{*1-\alpha}S_h^*(\tau-t)\|_{H\to H}d\tau\, \|v_h\|_{L_\infty[0,T;H]}$$

$$+\ ch^{-2q} \int_0^{k} \|S_h^*(\xi)\|_{H\to H}d\xi \int_{t+k}^{T} \|A_h^*S_h^*(\tau-t)\|_{H\to H}d\tau\, \|v_h\|_{L_\infty[0,T;H]}$$

$$+\ ch^{-2\bar{q}} \int_{(j-1)k}^{t} \|S_h^*(\tau-(j-1)k)\|_{H\to H}d\tau\, \|v_h\|_{L_\infty[0,T;H]}$$

By analyticity of $S_h^*(t)$, we have

$$\|e_2^*(t)\|_U \le ch^{-2\bar{q}} \int_0^k \frac{1}{\xi^\alpha} d\xi \int_t^{t+k} \frac{1}{(\tau-t)^{1-\alpha}} d\tau \|v_h\|_{L_\infty[0,T;H]}$$

$$+ ch^{-2\bar{q}} \int_0^k d\xi \int_{t+k}^{T} \frac{1}{\tau-t} d\tau \|v_h\|_{L_\infty[0,T;H]}$$

$$+ ch^{-2\bar{q}} \int_{(j-1)k}^{jk} d\tau \|v_h\|_{L_\infty[0,T;H]}$$

$$\le ch^{-2\bar{q}} k^{1-\alpha} k^\alpha \|v_h\|_{L_\infty[0,T;H]} + ch^{-2\bar{q}} k \ln \frac{T-t}{k} \|v_h\|_{L_\infty[0,T;H]}$$

$$+ ch^{-2\bar{q}} k \|v_h\|_{L_\infty[0,T;H]}$$

$$\le ch^{-2\bar{q}} k \ln \frac{1}{k} \|v_h\|_{L_\infty[0,T;H]}$$

As for case (ii),

$$\|e_2^*(t)\|_U \le ch^{-2\bar{q}} \Big\| \int_0^{t-(j-1)k} A_h^{*\alpha} S_h^*(\xi) d\xi \int_t^T A_h^{*1-\alpha} S_h^*(\tau-t) v_h(\tau) d\tau \Big\|_H$$

$$+ ch^{-2\bar{q}} \Big\| \int_{(j-1)k}^{t} S_h^*(\tau-(j-1)k) v_h(\tau) d\tau \Big\|_H$$

$$\le ch^{-2\bar{q}} \int_0^k \|A_h^{*\alpha} S_h^*(\xi)\|_{H\to H} d\xi \int_t^T \|A_h^{*1-\alpha} S_h^*(\tau-t)\|_{H\to H} d\tau \|v_h\|_{L_\infty[0,T;H]}$$

$$+ ch^{-2\bar{q}} \int_{(j-1)k}^{jk} \|S_h^*(\tau-(j-1)k)\|_{H\to H} d\tau \|v_h\|_{L_\infty[0,T;H]}$$

$$\leq ch^{-2\overline{q}} \int_0^k \frac{1}{\xi^\alpha} d\xi \int_t^T \frac{1}{(\tau-t)^{1-\alpha}} d\tau \|v_h\|_{L_\infty[0,T;H]}$$

$$+ ch^{-2\overline{q}} \int_{(j-1)k}^{jk} d\tau \|v_h\|_{L_\infty[0,T;H]}$$

$$\leq ch^{-2\overline{q}} k^{1-\alpha}(T-t)^\alpha \|v_h\|_{L_\infty[0,T;H]} + ch^{-2\overline{q}} k \|v_h\|_{L_\infty[0,T;H]}$$

$$\leq ch^{-2\overline{q}} k \|v_h\|_{L_\infty[0,T;H]}$$

Thus we have proved

$$\|e_2^*\|_{L_\infty[0,T;U]} \leq ch^{-2\overline{q}} k \ln \frac{1}{k} \|v_h\|_{L_\infty[0,T;H]}$$

$$\leq ch^{-2\overline{q}} k \ln \frac{1}{k} \|v\|_{L_\infty[0,T;H]}$$

For p = 1, we have

$$\int_0^T \|e_2^*(t)\|_U \, dt \leq \int_0^T \|B^*(S_h^*(t-(j-1)k)-I)\int_t^T S_h^*(\tau-t)v_h(\tau)d\tau\|_U dt$$

$$+ \int_0^T \|B^* \int_{(j-1)k}^t S_h^*(\tau-(j-1)k)v_h(\tau)d\tau\|_U dt$$

$$\leq ch^{-2\overline{q}} \int_0^T \int_t^T \|(S_h^*(t-(j-1)k)-I)S_h^*(\tau-t)v_h(\tau)\|_H d\tau dt$$

$$+ ch^{-2\overline{q}} \int_0^T \int_{(j-1)k}^t \|S_h^*(\tau-(j-1)k)v_h(\tau)\|_H d\tau dt$$

By reversing the order of integration,

$$\int_0^T \|e_2^*(t)\|_U dt \le ch^{-2\bar{q}} \int_0^T \int_0^\tau \Big\|\Big(\int_0^{t-(j-1)k} A_h^* S_h^*(\xi)d\xi\Big) S_h^*(\tau-t) v_h(\tau)\Big\|_H dt d\tau$$

$$+ ch^{-2\bar{q}} \int_0^T \int_\tau^{nk} \|S_h^*(\tau-(j-1)k)v_h(\tau)\|_H dt d\tau$$

where n is the unique integer such that $(n-1)k < \tau \le nk$. Furthermore,

$$\int_0^T \|e_2^*(t)\|_U dt \le ch^{-2\bar{q}} \int_0^k \int_0^\tau \Big\|\Big(\int_0^{t-(j-1)k} A_h^* S_h^*(\xi)d\xi\Big) S_h^*(\tau-t) v_h(\tau)\Big\|_H dt d\tau$$

$$+ ch^{-2\bar{q}} \int_k^T \int_0^{\tau-k} \Big\|\Big(\int_0^{t-(j-1)k} A_h^* S_h^*(\xi)d\xi\Big) S_h^*(\tau-t) v_h(\tau)\Big\|_H dt d\tau$$

$$+ ch^{-2q} \int_k^T \int_{\tau-k}^\tau \Big\|\Big(\int_0^{t-(j-1)k} A_h^* S_h^*(\xi)d\xi\Big) S_h^*(\tau-t) v_h(\tau)\Big\|_H dt d\tau$$

$$+ ch^{-2\bar{q}} \int_0^T \Big(\int_\tau^{nk} dt\Big) \|S_h^*(\tau-(j-1)k)v_h(\tau)\|_H d\tau$$

$$\le ch^{-2\bar{q}} \int_0^k \int_0^\tau \Big(\int_0^k \|A_h^{*\alpha} S_h^*(\xi)\|_{H\to H} d\xi\Big) \|A_h^{*1-\alpha} S_h^*(\tau-t)\|_{H\to H} dt \|v_h(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}} \int_k^T \int_0^{\tau-k} \Big(\int_0^k \|S_h^*(\xi)\|_{H\to H} d\xi\Big) \|A_h^* S_h^*(\tau-t)\|_{H\to H} dt \|v_h(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}} \int_k^T \int_{\tau-k}^\tau \Big(\int_0^k \|A_h^{*\alpha} S_h^*(\xi)\|_{H\to H} d\xi\Big) \|A_h^{*1-\alpha} S_h^*(\tau-t)\|_{H\to H} dt \|v_h(\tau)\|_H d\tau$$

$$+ \; ch^{-2\bar{q}}k \int_0^T \|S_h^*(\tau-(j-1)k)\|_{H\to H} \|v_h(\tau)\|_H d\tau$$

$$\leq ch^{-2\bar{q}} \int_0^k k^{1-\alpha}\left(\int_0^\tau \frac{1}{(\tau-t)^{1-\alpha}}\, dt\right) \|v_h(\tau)\|_H d\tau$$

$$+ \; ch^{-2\bar{q}} \int_k^T k\left(\int_0^{\tau-k} \frac{1}{\tau-t}\, dt\right) \|v_h(\tau)\|_H d\tau$$

$$+ \; ch^{-2\bar{q}} \int_k^T k^{1-\alpha}\left(\int_{\tau-k}^\tau \frac{1}{(\tau-t)^{1-\alpha}}\, dt\right) \|v_h(\tau)\|_H d\tau$$

$$+ \; ch^{-2\bar{q}}k \int_0^T \|v_h(\tau)\|_H d\tau$$

$$\leq ch^{-2\bar{q}}k \|v_h\|_{L_1[0,T;H]} + ch^{-2\bar{q}}k \ln \frac{1}{k} \|v_h\|_{L_1[0,T;H]}$$

$$+ \; ch^{-2\bar{q}}k \|v_h\|_{L_1[0,T;H]} + ch^{-2\bar{q}}k \|v_h\|_{L_1[0,T;H]}$$

$$\leq ch^{-2\bar{q}}k \ln \frac{1}{k} \|v_h\|_{L_1[0,T;H]}$$

Thus we have proved

$$\|e_2^*\|_{L_1[0,T;U]} \leq ch^{-2\bar{q}}k \ln \frac{1}{k} \|v_h\|_{L_1[0,T;H]}.$$

We now undertake the estimation of $e_3^*(t) = T_h^* v((j-1)k) - T_{hk}^* v(jk)$, which we decompose in the following way:

$$e_3^*(t) = B^*\Big[\int_{(j-1)k}^{T} S_h^*(\tau-(j-1)k)P_h v(\tau)d\tau - k\sum_{n=j}^{N} E_{hk}^{*n+1-j} P_h(P_k v)(nk)\Big]$$

$$= B^*\sum_{n=j}^{N}\Big[\int_{(n-1)k}^{nk} S_h^*(\tau-(j-1)k)P_h v(\tau)d\tau - kE_{hk}^{*n+1-j} P_h(P_k v)(nk)\Big].$$

As in the estimation of $e_2^*(t)$, we shall assume without loss of generality that $v(t) = P_h v(t) = v_h(t)$. We also make the following observation, based on the definition of P_k:

$$\int_{(n-1)k}^{nk} S_h^*((n+1-j)k)v_h(\tau)d\tau = S_h^*((n+1-j)k)\int_{(n-1)k}^{nk} v_h(\tau)d\tau$$

$$= kS_h^*((n+1-j)k)(P_k v_h)(nk), \quad n=1,2,\ldots \tag{6.5.5}$$

Hence we have the further decomposition

$$e_3^*(t) = B^*\sum_{n=j}^{N}\Big[\int_{(n-1)k}^{nk} S_h^*(\tau-(j-1)k)v_h(\tau)d\tau - \int_{(n-1)k}^{nk} S_h^*((n+1-j)k)v_h(\tau)d\tau\Big]$$

$$+ B^*\sum_{n=j}^{N}\Big[kS_h^*((n+1-j)k)(P_k v_h)(nk) - kE_{hk}^{*n+1-j}(P_k v_h)(nk)\Big]$$

$$= B^*\sum_{n=j}^{N}\int_{(n-1)k}^{nk} (S_h^*(\tau-(j-1)k) - S_h^*((n+1-j)k))v_h(\tau)d\tau$$

$$+ B^*\sum_{n=j}^{N} k(S_h^*((n+1-j)k) - E_{hk}^{*n+1-j})(P_k v_h)(nk)$$

By application of (6.5.3), we have

$$|e_3^*(t)|_U \leq ch^{-2\bar{q}} \sum_{n=j}^{N} \Big| \int_{(n-1)k}^{nk} (S_h^*(\tau-(j-1)k)-S_h^*((n+1-j)k))v_h(\tau)d\tau \Big|_H$$

$$+ \; ch^{-2\bar{q}} \sum_{n=j}^{N} k \big|(S_h^*((n+1-j)k)-E_{hk}^{*n+1-j})(P_k v_h)(nk)\big|_H$$

$$\leq ch^{-2\bar{q}} \int_{(j-1)k}^{jk} (|S_h^*(\tau-(j-1)k)|_{H\to H} + |S_h^*((n+1-j)k)|_{H\to H})|v_h(\tau)|_H d\tau$$

(†) $$+ \; ch^{-2\bar{q}} \sum_{n=j+1}^{N} \int_{(n-1)k}^{nk} |(S_h^*(\tau-(j-1)k)-S_h^*((n+1-j)k)|_{H\to H}|v_h(\tau)|_H d\tau$$

$$+ \; ch^{-2\bar{q}}k \sum_{n=j}^{N} |S_h^*((n+1-j)k)-E_{hk}^{*n+1-j}|_{H\to H}|(P_k v_h)(nk)|_H$$

$$\leq ch^{-2\bar{q}}k\Big(\frac{1}{k}\int_{(j-1)k}^{jk} |v_h(\tau)|_H d\tau\Big)$$

(†) $$+ \; ch^{-2\bar{q}} \sum_{n=j+1}^{N} \int_{(n-1)k}^{nk} \Big|\int_{\tau-(j-1)k}^{(n+1-j)k} A_h^*S_h^*(\xi)d\xi\Big|_{H\to H}|v_h(\tau)|_H d\tau$$

$$+ \; ch^{-2\bar{q}}k \sum_{n=j}^{N} |S_h^*((n+1-j)k)-E_{hk}^{*n+1-j}|_{H\to H}|(P_k v_h)(nk)|_H$$

(†) This term is absent if j=N. (Same convention on next page).

By definition of P_k, by analyticity of $S_h^*(t)$, and by Lemma 4.1, it follows that

$$|e_3^*(t)|_U \le ch^{-2\bar{q}}\, k\Big(\frac{1}{k}\int_{(j-1)k}^{jk} |v_h(\tau)|_H d\tau\Big)$$

$$(\dagger) \qquad + ch^{-2\bar{q}} \sum_{n=j+1}^{N} \int_{(n-1)k}^{nk} \Big(\int_{(n-j)k}^{(n+1-j)k} \frac{1}{\xi} d\xi\Big) |v_h(\tau)|_H d\tau$$

$$+ ch^{-2\bar{q}}\, k \sum_{n=j}^{N} \frac{1}{n+1-j} |(P_k v_h)(nk)|_H$$

$$\le ch^{-2\bar{q}}\, k\Big(\frac{1}{k}\int_{(j-1)k}^{jk} |v_h(\tau)|_H d\tau\Big)$$

$$(\dagger) \qquad + ch^{-2\bar{q}} \sum_{n=j+1}^{N} \ln \frac{n+1-j}{n-j} \int_{(n-1)k}^{nk} |v_h(\tau)|_H d\tau$$

$$+ ch^{-2\bar{q}}\, k \sum_{n=j}^{N} \frac{1}{n+1-j} |(P_k v_h)(nk)|_H$$

For $p = \infty$, we have, since $e_3^*(t) = e_3^*(jk)$,

$$\begin{aligned} |e_3^*|_{L_\infty[0,T;U]} = \max_{1\le j\le N} |e_3^*(jk)|_U &\le ch^{-2\bar{q}} k |v_h|_{L_\infty[0,T;H]} \\ &+ ch^{-2\bar{q}} k \ln N |v_h|_{L_\infty[0,T;H]} \\ &+ ch^{-2\bar{q}} k \ln N |P_k v_h|_{L_\infty[0,T;H]} \end{aligned}$$

Thus we have proved

$$|e_3^*|_{L_\infty[0,T;U]} \le ch^{-2\bar{q}} k \ln \frac{1}{k} |v_h|_{L_\infty[0,T;H]}$$

$$\le ch^{-2\bar{q}}k \ln \frac{1}{k}\|v\|_{L_\infty[0,T;H]}.$$

Similarly, for p = 1, we have

$$\|e_3^*\|_{L_1[0,T;U]} = k\sum_{j=1}^{N}\|e_3^*(jk)\|_U \le ch^{-2\bar{q}}k\sum_{j=1}^{N}\int_{(j-1)k}^{jk}\|v_h(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}}k\sum_{j=1}^{N-1}\sum_{n=j+1}^{N}\ln\frac{n+1-j}{n-j}\int_{(n-1)k}^{nk}\|v_h(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}}k^2\sum_{j=1}^{N}\sum_{n=j}^{N}\frac{1}{n+1-j}\|(P_k v_h)(nk)\|_H$$

By reversing the order of summation,

$$\|e_3^*\|_{L_1[0,T;U]} \le ch^{-2\bar{q}}k\|v_h\|_{L_1[0,T;H]}$$

$$+ ch^{-2\bar{q}}k\sum_{n=2}^{N}\Big(\sum_{j=1}^{n-1}\ln\frac{n+1-j}{n-j}\Big)\int_{(n-1)k}^{nk}\|v_h(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}}k^2\sum_{n=1}^{N}\Big(\sum_{j=1}^{n}\frac{1}{n+1-j}\Big)\|(P_k v_h)(nk)\|_H$$

$$\le ch^{-2\bar{q}}k\|v_h\|_{L_1[0,T;H]}$$

$$+ ch^{-2\bar{q}}k\sum_{n=2}^{N}\ln n\int_{(n-1)k}^{nk}\|v_h(\tau)\|_H d\tau$$

$$+ ch^{-2\bar{q}}k^2 \sum_{n=1}^{N} \ln n \|(P_k v_h)(nk)\|_H$$

$$\le ch^{-2\bar{q}}k \|v_h\|_{L_1[0,T;H]} + ch^{-2\bar{q}}k \ln N \|v_h\|_{L_1[0,T;H]}$$

$$+ ch^{-2\bar{q}}k \ln N (k \sum_{n=1}^{N} \|(P_k v_h)(nk)\|_H)$$

$$\le ch^{-2\bar{q}}k \|v_h\|_{L_1[0,T;H]} + ch^{-2\bar{q}}k \ln \frac{1}{k} \|v_h\|_{L_1[0,T;H]}$$

Thus we have proved

$$\|e_3^*\|_{L_1[0,T;U]} \le ch^{-2\bar{q}}k \ln \frac{1}{k} \|v_h\|_{L_1[0,T;H]}$$

$$\le ch^{-2\bar{q}}k \ln \frac{1}{k} \|v\|_{L_1[0,T;H]}$$

Since $(T^*-T_{hk}^*)v(t) = e_1^*(t) + e_2^*(t) + e_3^*(t)$, this concludes the proof of Theorem 5.1.

6.6 Proof of Theorem 5.2

Before starting with the proof, we give a corollary of Theorem 4.2.

Corollary 6.3. Let T^* be defined as in (2.37), and let $\tilde{T}^*$ be defined by

$$\tilde{T}^*v(t) = \tilde{T}^*v(jk) = B^* \sum_{n=j}^{N} \int_{(n-1)k}^{nk} S^*((n+1-j)k)(P_k v)(\tau)d\tau.$$

Then

$$(6.6.1) \qquad |(T^*-\tilde{T}^*)v|_{L_p[0,T;U)} \le ck^{1-q-\varepsilon}|v|_{L_p[0,T;H]}, \quad p = 1, \infty.$$

//

Proof of Corollary 6.3. The result follows immediately from the observation that $(T^*-\tilde{T}^*)v(t) = e_1^*(t) + e_2^*(t) + e_3^*(t) + e_4^*(t)$, with $e_i^*(t)$ as defined in the proof of Theorem 4.2. In particular, by taking $\alpha = 0$ in Theorem 4.2 one obtains (6.6.1).

//

Coming back to the proof of Theorem 5.2, we set $(T^*-T_{hk}^*)v(t) = e_1^*(t) + e_2^*(t) + e_3^*(t)$, where

$$e_1^*(t) = (T^* - \tilde{T}^*)v(t),$$

$$e_2^*(t) = (\tilde{T}^* - \tilde{T}_h^*)v(t),$$

$$e_3^*(t) = (\tilde{T}_h^* - T_{hk}^*)v(t),$$

with $\tilde{T}_h^*v(t) = B^* \sum_{n=j}^{N} \int_{(n-1)k}^{nk} S_h^*((n+1-j)k)P_h(P_k v)(\tau)d\tau.$

The estimation of $e_1^*(t)$ is precisely the conclusion of Corollary 6.3, i.e., (6.6.1). The estimation of the other two terms $e_2^*(t)$ and $e_3^*(t)$ will require the following result.

Lemma 6.4. For $0 < h < 1$, let $\{V_h\}$ be a family of subspaces of $D(A^*)$ for which properties (3.1) and (3.4') hold, and let $\{A_h\}$ be a family of approximations of A such that properties (3.5), (3.6) are satisfied for both A and A^*. Then

$$A^*A_h^{*-1}P_h \in \mathscr{L}(H\to H)$$

//

Proof of Lemma 6.4. For simplicity let $v_h = P_hv$ for $v \in H$. We have

$$\|A^*A_h^{*-1}v_h\|_H = \|A_h^{*-1}v_h\|_{D(A^*)}$$

$$\leq \|(P_hA^{*-1} - A_h^{*-1}P_h)v_h\|_{D(A^*)} + \|P_hA^{*-1}v_h\|_{D(A^*)}$$

Using the inverse assumption (3.4'), we derive

$$\|A^*A_h^{*-1}v_h\|_H \leq ch^{-2}\|(P_hA^{*-1}-A_h^{*-1}P_h)v_h\|_H + c\|v_h\|_H$$

$$\leq ch^{-2}[\|(A^{*-1}-A_h^{*-1}P_h)v_h\|_H + \|(I-P_h)A^{*-1}v_h\|_H+h^2\|v_h\|_H]$$

We now use (3.1) and (3.5) to obtain

$$\|A^*A_h^{*-1}v_h\|_H \leq ch^{-2}[ch^2\|v_h\|_H+ ch^2\|v_h\|_H + h^2\|v_h\|_H]$$

$$\leq c\|v_h\|_H$$

and thus Lemma 6.4 is proved.

//

Coming back to the proof of Theorem 5.2, we are now in a position to estimate $e_2^*(t)$ and $e_3^*(t)$.

From the observation (6.5.5) we made in section 6.5, we have

$$(\tilde{T}^* - \tilde{T}_h^*)v(t) = (\tilde{T}^* - \tilde{T}_h^*)v(jk)$$

$$= B^* \sum_{n=j}^{N} \int_{(n-1)k}^{nk} [S^*((n+1-j)k)-S_h^*((n+1-j)k)P_h](P_k v)(\tau)d\tau$$

$$= B^* \sum_{n=j}^{N} k(S^*((n+1-j)k)-S_h^*((n+1-j)k)P_h)(P_k v)(nk).$$

Thus

$$|e_2^*(t)|_U = |B^*A^{*-q} \sum_{n=j}^{N} kA^{*q}(S^*((n+1-j)k)-S_h^*((n+1-j)k)P_h)(P_k v)(nk)|_U$$

$$\leq c \sum_{n=j}^{N} k|A^{*q}(S^*((n+1-j)k)-S_h^*((n+1-j)k)P_h)|_{H\to H}|P_k v(nk)|_H.$$

We recall from Lasiecka [11] that

$$(6.6.2) \qquad |S^*(t)-S_h^*(t)P_h|_{H\to H} \leq c \frac{h^{2\ell}}{t^{\ell}} \quad , 0 \leq \ell \leq 1, \ t > 0.$$

By Lemma 6.4 and by analyticity of $S^*(t)$ and $S_h^*(t)$, we also have that

$$|A^*(S^*(t)-S_h^*(t)P_h)|_{H\to H} \leq |A^*S^*(t)|_{H\to H} + |A^*A_h^{*-1}|_{H\to H}|A_h^*S_h^*(t)P_h|_{H\to H}$$

$$(6.6.3) \qquad \leq \frac{c}{t}, \qquad t > 0.$$

We now apply the same interpolation theorem [13] we have used several times before to (6.6.2) and (6.6.3). Thus it follows that

$$\|A^{*q}(S^*(t) - S_h^*(t)P_h)\|_{H\to H} \le c\, \frac{h^{2\ell(1-q)}}{t^{\ell(1-q)}} \cdot \frac{1}{t^q} \quad , \; 0 \le q \le 1.$$

Hence by taking $\ell = 1 - \frac{\varepsilon}{1-q}$,

$$\|e_2^*(t)\|_U \le c \sum_{n=j}^{N} k \frac{h^{2(1-q-\varepsilon)}}{((n+1-j)k)^{1-\varepsilon}} \|(P_k v)(nk)\|_H.$$

For $p = \infty$, we have

$$\begin{aligned} \|e_2^*\|_{L_\infty[0,T;U]} &= \max_{1 \le j \le N} \|e_2^*(jk)\|_U \\ &\le ch^{2(1-q-\varepsilon)} \Big(\sum_{n=1}^{N} k^{\varepsilon} \cdot \frac{1}{n^{1-\varepsilon}} \Big) \|P_k v\|_{L_\infty[0,T;H]} \\ &\le ch^{2(1-q-\varepsilon)} \|v\|_{L_\infty[0,T;H]} \end{aligned}$$

For $p = 1$, we have similarly

$$\begin{aligned} \|e_2^*\|_{L_1[0,T;U]} &= k \sum_{j=1}^{N} \|e_2^*(jk)\|_U \\ &\le ckh^{2(1-q-\varepsilon)} \sum_{j=1}^{N} \sum_{n=j}^{N} k^{\varepsilon} \cdot \frac{1}{(n+1-j)^{1-\varepsilon}} \|(P_k v)(nk)\|_H \\ &\le ckh^{2(1-q-\varepsilon)} \sum_{n=1}^{N} \Big(\sum_{j=1}^{n} k^{\varepsilon} \cdot \frac{1}{(n+1-j)^{1-\varepsilon}} \Big) \|(P_k v)(nk)\|_H \\ &\le ckh^{2(1-q-\varepsilon)} \sum_{n=1}^{N} (nk)^{\varepsilon} \|(P_k v)(nk)\|_H \end{aligned}$$

$$\leq ch^{2(1-q-\varepsilon)}\left(k \sum_{n=1}^{N} |(P_k v)(nk)|_H\right)$$

$$\leq ch^{2(1-q-\varepsilon)} |v|_{L_1[0,T;H]}$$

The estimation of e_2^* is completed, and now we turn to

$$e_3^*(t) = (\tilde{T}_h^* - T_{hk}^*)v(t)$$

$$= B^* \sum_{n=j}^{N} k(S_h^*((n+1-j)k) - E_{hk}^{*n+1-j})P_h(P_k v)(nk)$$

In fact, by (6.5.3),

$$|e_3^*(t)|_U = |B^* A^{*-q} \sum_{n=j}^{N} kA^{*q}(S_h^*((n+1-j)k) - E_{hk}^{*n+1-j})P_h(P_k v)(nk)|_U$$

$$\leq c \sum_{n=j}^{N} k |A^{*q}(S_h^*((n+1-j)k) - E_{hk}^{*n+1-j})P_h|_{H\to H} |(P_k v)(nk)|_H .$$

From Lemma 4.1 applied to the semigroup $S_h^*(t)$, we have

$$(6.6.4) \qquad |(S_h^*((n+1-j)k) - E_{hk}^{*n+1-j})P_h|_{H\to H} \leq \frac{c}{n+1-j}$$

and

$$(6.6.5) \qquad |A_h^*(S_h^*((n+1-j)k) - E_{hk}^{*n+1-j})P_h|_{H\to H} \leq \frac{c}{(n+1-j)k} .$$

Hence by Lemma 6.4,

$$(6.6.6) \qquad |A^*(S_h^*((n+1-j)k) - E_{hk}^{*n+1-j})P_h|_{H\to H} \leq \frac{c}{(n+1-j)k} .$$

Interpolation between (6.6.4) and (6.6.6) yields

$$\|A^{*q}(S_h^*((n+1-j)k)-E_{hk}^{*n+1-j})P_h\|_{H\to H} \le \frac{c}{(n+1-j)} \cdot \frac{1}{k^q} \quad , 0 \le q \le 1.$$

Therefore,

$$\|e_3^*(t)\|_U \le c \sum_{n=j}^{N} k^{1-q-\varepsilon}.k^{\varepsilon} \frac{1}{(n+1-j)}\|(P_k v)(nk)\|_H \quad , \quad \varepsilon>0.$$

For p = ∞,

$$\|e_3^*\|_{L_\infty[0,T;U]} \le ck^{1-q-\varepsilon}(\sum_{n=1}^{N} k^{\varepsilon}. \frac{1}{n})\|P_k v\|_{L_\infty[0,T;H]}$$

$$\le ck^{1-q-\varepsilon}\|v\|_{L_\infty[0,T;H]}.$$

Similarly, for p = 1,

$$\|e_3^*\|_{L_1[0,T;U]} \le ck^{1-q-\varepsilon}k \sum_{j=1}^{N} \sum_{n=j}^{N} k^{\varepsilon}\frac{1}{n+1-j}\|(P_k \text{[illegible]} (nk)\|_H$$

$$\le ck^{1-q-\varepsilon}k \sum_{n=1}^{N} (\sum_{j=1}^{n} k^{\varepsilon}\frac{1}{n+1-j})\|\text{[illegible]} (nk)\|_H$$

$$\le ck^{1-q-\varepsilon}k \sum_{n=1}^{N} \|(P_k v)(nk)\|_H$$

$$\le ck^{1-q-\varepsilon}\|v\|_{L_1[0,T;H]}$$

Since $(T^*-T^*_{hk})v(t) = e^*_1(t)$, $e^*_2(t) + e^*_3(t)$, the proof of Theorem 5.2 is complete.

CHAPTER VII
NUMERICAL EXAMPLES

7.1 Opening Remarks

In this chapter we shall implement the fully discrete approximation scheme described in Chapter V in the canonical case of the heat equation. We shall also present numerical results from a computer program which implements this discrete scheme in the one-dimensional case with $\Omega = (0,1)$. These results will be presented in the form of tables and figures, and we shall moreover explain the nature and scope of these results so as to facilitate their interpretation

7.2 Implementation of the Fully Discrete Scheme for the Heat Equation

Consider the canonical heat equation with nonhomogeneous Dirichlet boundary conditions

$$(7.1)\begin{cases} u_t(t,x) = \Delta u(t,x) & \text{on } Q = (0,T)x\Omega \\ u(0,x) = u_o(x) & \text{on } \Omega \\ u|_\Gamma = g & \text{on } \Sigma = (0,T)x\Gamma \end{cases}$$

Let $A = -\Delta$ + zero Dirichlet boundary conditions. Then by Green's formula, the bilinear form $a(u,v)$ associated to $-A$ is

$$a(u,v) = (\Delta u,v)_\Omega = -(\nabla u,\nabla v)_\Omega + \langle\frac{\partial u}{\partial\nu},v\rangle_\Gamma$$

$$= -(\nabla u,\nabla v)_\Omega \qquad \forall u,v \in \overset{\circ}{H}_1(\Omega).$$

We now recall from Chapter III the continuous-time approximation algorithm:

find $u_h(t) \in V_h$ such that

$$(7.2)\quad \begin{cases} (\dot{u}_h(t),v_h)_\Omega = -(\nabla u_h(t),\nabla v_h)_\Omega - \langle g(t), \frac{\partial v_h}{\partial\nu}\rangle_\Gamma \\ u_h(0) \text{ given} \end{cases} .$$

If $\{\phi_i\}_{i=1}^M$ form a basis for the subspace V_h, and if we let

$u_h(t) = \sum_{i=1}^M a_i(t)\phi_i$, then (7.2) can be reformulated as

$$(7.3)\quad \begin{cases} \sum_{i=1}^M \dot{a}_i(t)(\phi_i,\phi_j)_\Omega = -\sum_{i=1}^M a_i(t)(\nabla\phi_i,\nabla\phi_j)_\Omega - \langle g(t),\frac{\partial\phi_j}{\partial\nu}\rangle_\Gamma, \; j=1,\dots,M \\ a_i(0) \text{ given}, \; i=1,\dots,M. \end{cases}$$

Let $a(t) = (a_1(t),\dots,a_M(t))^T$, $L = ((\phi_i,\phi_j)_\Omega)_{i,j=1,\dots,M}$,

$K = ((\nabla\phi_i,\nabla\phi_j)_\Omega)_{i,j=1,\ldots,M}$, and $F = (\frac{\partial\phi_1}{\partial\nu},\ldots,\frac{\partial\phi_M}{\partial\nu})^T$. Then (7.3) has the following compact form:

$$(7.4)\quad \begin{cases} L\dot{a}(t) = -Ka(t) - \langle g(t),F\rangle_\Gamma \\ a(0) \text{ given} \end{cases}$$

We are dealing here with a linear system of ordinary differential equations. Thus the discrete-time approximating scheme described in section 5.2 can be applied to (7.4) with minimal modifications. If we set $g_n = g(nk)$, where k is the parameter of discretization in time, and let a_n be the discrete-time approximation of a(nk), then the fully discrete approximation of (7.1) takes the form

$$\begin{cases} \frac{1}{k} L(a_{n+1}-a_n) = -Ka_{n+1}-\langle g_{n+1},F\rangle_\Gamma \qquad n=0,1,\ldots \\ a_o = a(0) \text{ given} \end{cases}$$

or by factoring a_{n+1},

$$(7.5)\quad \begin{cases} (L+kK)a_{n+1} = La_n - k\langle g_{n+1},F\rangle_\Gamma, \qquad n=0,1,\ldots \\ a_o = a(0) \text{ given} \end{cases}$$

As we pointed out in Chapter V, we are now faced with a sequence of linear systems of algebraic equations. This is theoretically a simple task, but in practice it can be quite challenging for very large values of M. The difficulty also increases with the condition number of L + kK, which itself is usually difficult to evaluate.

7.3 Numerical Computations and Results

In this section we shall calculate the matrices L and K as well as the vector F in the case of the one-dimensional heat equation with $\Omega = (0,1)$, and where the approximating functions are linear splines.

More precisely, if the unit interval is partitioned into $M = 1/h$ equal subintervals, then the "hat" functions ϕ_i defined by

$$\phi_i(x) = \begin{cases} \frac{1}{h}x - (i-1) & \text{if } (i-1)h \leq x \leq ih \\ \frac{-1}{h}x + (i-1) & \text{if } ih \leq x \leq (i+1)h, \quad i=1,\dots,M-1 \\ 0 & \text{otherwise} \end{cases}$$

span a space of linear splines on [0,1] of dimension M-1. We note that these splines satisfy zero Dirichlet boundary conditions.

It can be readily seen that

$$(\phi_i,\phi_{i-1})_\Omega = \frac{1}{6}h \qquad \text{for } i = 2,\dots,M-1$$

and that

$$(\phi_i,\phi_i)_\Omega = \frac{2}{3}h \qquad \text{for } i=1,\dots,M-1$$

Also $\quad (\phi_i,\phi_j)_\Omega = 0 \qquad \text{for } |i-j| \geq 2$

Furthermore

$$(\nabla\phi_i,\nabla\phi_{i-1})_\Omega = -\frac{1}{h}, \qquad i = 2,\dots,M-1$$

and

$$(\nabla\phi_i,\nabla\phi_i)_\Omega = \frac{2}{h}, \qquad i=1,\dots,M-1$$

and also

$$(\nabla\phi_i,\nabla\phi_j)_\Omega = 0 \qquad \text{if } |i-j| \geq 2.$$

Finally, we have

$$\frac{\partial\phi_i}{\partial\nu} = \begin{cases} -\frac{1}{h} & \text{if } i=1,M-1 \\ 0 & \text{if } i\neq 1,M-1 \end{cases}$$

Remark. We can see that the matrices L and K are both tridiagonal. This property is very advantageous from a computational point of view.

//

Thus, after simplification, the fully discrete approximation (7.5) takes the form

$$(7.6)\quad \begin{cases} \Psi a_{n+1} = \Phi a_n + cG, & n = 0,1,\dots \\ a_o \text{ given} \end{cases}$$

where $c = 6\,\frac{k}{h^2}$, $\Psi_{i,i} = 4 + 2c$, $\Psi_{i,i-1} = \Psi_{i-1,i} = 1-c$,

$\Phi_{i,i} = 4$, $\Phi_{i,i-1} = \Phi_{i-1,i} = 1$, $G(1) = g_{n+1}(0)$, $G(M-1) = g_{n+1}(1)$,

and all other entries are equal to zero.

We shall now present the results of a computer program which solves the sequence of linear systems (7.6) for given input data a_o and $\{g_n\}$ such that the formula for the corresponding solution of problem (7.1) is known. The program then calculates the relative error of discretization e_{hk} in the norms of $L_1[0,T;L_2(\Omega)]$ (resp. $L_\infty[0,T;L_2(\Omega)]$) for g in $L_1[0,T;L_2(\Gamma)]$ (resp. $L_\infty[0,T;L_2(\Gamma)]$), i.e.,

$$e_{hk} = \frac{|u-u_{hk}|_{L_p[0,T;L_2(\Omega)]}}{|g|_{L_p[0,T;L_2(\Gamma)]}}, \qquad p = 1, \infty,$$

The values of e_{hk} as a function of h and k are calculated for two particular solutions, say u_1 and u_2, where

$$u_1(t,x) = x(x-1)[\frac{x^2-x-1}{12} + t] + t^2 \qquad , t > 0, 0 \leq x \leq 1,$$

and

$$u_2(t,x) = e^{x+t(1-\pi^2)} \cos \pi(x+2t) \qquad , t > 0, 0 \leq x \leq 1.$$

In tables 1 and 2, the values of e_{hk} corresponding to u_1 are shown in both the cases $p = 1$ and $p = \infty$ respectively; the same is true for tables 3 and 4 where this time e_{hk} corresponds to u_2. More precisely, the values of $\log_{10}(10^2 e_{hk})$ are displayed in terms of $\log_2 \frac{1}{h}$ and $\log_2 \frac{1}{k}$, which increase by 1 whenever h and k are divided by 2.

If we let $k = ch^2$ for some $c > 0$, then $h^{1/2} + k^{1/4} = h^{1/2}(1 + c^{1/4})$.

So

$$\log_{10}(h^{1/2} + k^{1/4}) = \log_{10} h^{1/2} + \log_{10} (1 + c^{1/4})$$

$$= - 1/2 \log_{10} \frac{1}{h} + c$$

$$= - 1/2 .(\log_{10}2)(\log_2 \frac{1}{h}) + c$$

Thus the rate of decrease of $\log_{10}(h^{1/2} + k^{1/4})$ with respect to $\log_2 \frac{1}{h}$ is constant. Moreover, we can see that the observed rate of -0.15 (asymptotic rate) is very close to the theoretical value of $-\frac{1}{2}\log_{10} 2 = -0.1505$. Our theoretical rates of convergence are therefore confirmed by these numerical examples.

The information contained in tables 1-4 is displayed in graphic fashion in figures 1-4. These figures provide a more intuitive and qualitative look at the numerical results. For example, the figures clearly indicate that the theoretical restriction $k \leq ch^2$ is also necessary in practice in order to obtain the best rate of convergence. It is also apparent from figures 3 and 4 that the strategy of fixing one parameter of discretization while decreasing the other is not only suboptimal, but can also cause serious instability problems.

The theoretical rates of convergence derived in Theorems 5.3 and 5.4 are valid for "rough" boundary input, i.e., for input in $L_p[0,T;L_2(\Gamma)]$. In fact, our last example shows calculated values of the solution in the case of discontinuous boundary input. These values are calculated for two choices of the boundary input as the mesh size $h = \frac{1}{M}$ is decreased while respecting the condition $k = ch^2$. As can be seen in Table 5, the calculated values of both solutions exhibit a clearly convergent behavior. This confirms the stability of our scheme (5.7).

Remark. The solutions u_1 and u_2 restricted to the boundary are clearly infinitely differentiable. It is also clear, however, that the rates of convergence are not improved although one might expect this to happen. An alternative to this situation may be provided by an extension of Nitsche's method. In fact, in the semidiscrete case it is known that Nitsche's method [17] has rates of convergence which improve with the smoothness of the boundary data.

//

REFERENCES

[1] I. BABUSKA, The finite element method with lagrangian multipliers, Numer. Math., Vol. 20, 1973, pp. 179-192.

[2] I. BABUSKA and A. AZIZ, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, Academic Press, New York, 1972.

[3] A. V. BALAKRISHNAN, Applied Functional Analysis, Springer-Verlag, Berlin, 1976.

[4] J. BRAMBLE, A. SCHATZ, V. THOMÉE and L. WAHLBIN, Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations, SIAM J. of Numer. Anal., Vol. 14, no. 2, 1977, pp. 218-241.

[5] J. DOUGLAS and T. DUPONT, Galerkin methods for parabolic equations, SIAM J. of Numer. Anal., Vol. 4, 1970, pp. 575-626.

[6] N. DUNFORD and I. SCHWARTZ, Linear Operators, Vol. II, Interscience, New York, 1963.

[7] D. FUJIWARA, Concrete characterizations of the domains of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad., Vol. 43 (1967), pp. 82-86.

[8] M. HUANG and V. THOMEE, Some convergence estimates for semidiscrete type schemes for time dependent nonselfadjoint parabolic equations, Math. Comp. Vol. 37, 1981, pp. 327-346.

[9] M. HUANG and V. THOMEE, On the backward Euler method for parabolic equations with rough initial data, SIAM J. of Numer. Anal., Vol. 19, 1982, pp. 599-603.

[10] I. LASIECKA, A unified theory for abstract parabolic boundary problems - a semigroup approach, Appl. Math. Optimization, Vol. 6, 1980, pp. 287-333.

[11] I. LASIECKA, Convergence estimates for semidiscrete approximations of nonselfadjoint parabolic equations, SIAM J. of Numer. Anal., Vol. 21, 1984, pp. 894-909.

[12] I. LASIECKA, Galerkin approximations of abstract parabolic boundary-value problems with rough boundary data - L_p theory, Math. Comp., Vol. 47, 1986, pp. 55-75.

[13] J. L. LIONS and E. MAGENES, Nonhomogeneous Boundary Value Problems and Applications, Vol. I, II, Springer-Verlag, New York, 1972.

[14] M. LUSKIN and R. RANNACHER, On the smoothing of the Galerkin method for parabolic equations, SIAM J. of Numer. Anal., Vol. 19, 1981, pp. 93-113.

[15] U. MACKENROTH, Numerical solution of some parabolic boundary control problems by finite element methods, Lect. Notes in Control & Inf. Sc., Vol. 97, Springer-Verlag, New York, 1987, pp. 325-336.

[16] J. NECAS, Les Méthodes Directes en Theorie des Equations Elliptiques, Editions Academie Czechoslovaque des Sciences, Prague, 1967.

[17] J. NITSCHE, Über ein Variations prinzip zür Lösung von Dirichlet Problem bei Verwendung von Teilraümen, die keinen Randbedingungen unterworfen sind, Abh. Math. Sem. Univ. Hamburg, Vol. 36, 1971, pp. 9-15.

[18] A. PAZY, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., Springer-Verlag, New York, 44, 1983.

[19] P. H. SAMMON, Convergence estimates for semidiscrete parabolic equation approximations, SIAM J. Numer. Anal., Vol. 19, 1982, pp. 68-81.

[20] E. STEIN, Singular integrals, harmonic functions and differentiability properties of functions of several variables, Proc. Sympos. Pure Math., Vol. 10, Amer. Math. Soc., Providence, RI, 1967.

[21] R. WINTHER, A numerical Galerkin method for a parabolic control problem, Ph.D. Thesis, Cornell University, 1977.

BIOGRAPHICAL SKETCH

The author was born in Karachi, Pakistan, on November 14, 1957. He grew up in France with a traditional education. He came to the United States in 1979 and completed his master's degree in 1981. He was a graduate student at the University of Florida from 1979 to 1987.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. Irena Lasiecka
Chairman

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. Bruce Edwards
Assoc. Chair & Assoc. Prof. of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

V. M. Popov

Dr. Vasile Popov
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

N. Dinculeanu

Dr. Nicolae Dinculeanu
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. Pierre Sikivie
Assoc. Professor of Physics

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts & Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Dr. Madelyn Lockhart
Dean of Graduate School

December 1987